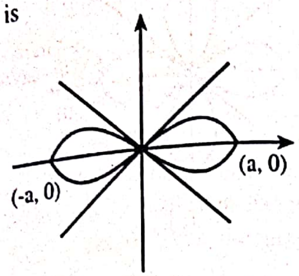
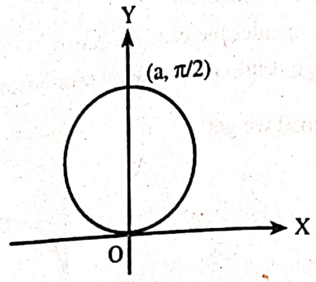
ii) The graph of the curve $r^2 = a^2 \cos^2 2\theta$ is



iii) The graph of the curve $r = a \sin\theta$ is



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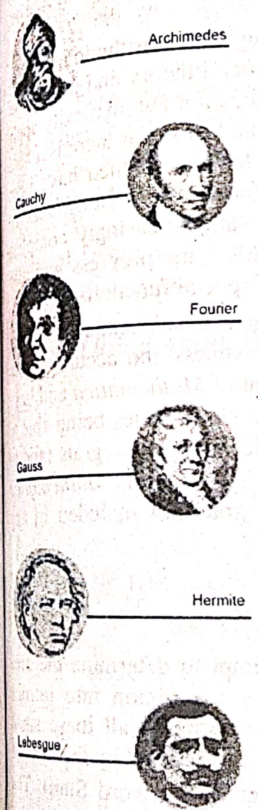
Chapter - 11

INTEGRAL CALCULUS

History of Integration

Mathematica combines the recent decades of the computer revolution with the previous few centuries of mathematical research, fulfilling the original goal of the early computer pioneers: to do mathematics by computer.



Over 2000 years ago, Archimedes (287-312 BC) found formulas for the surface areas and volumes of solids such as the sphere, the cone, and the paraboloid. His method of integration was remarkably modern considering that he did not have algebra, the function concept, or even the decimal representation of numbers.

Leibniz (1646-1716) and Newton (1642-1727) independently discovered calculus. Their key idea was that differentiation and integration undo each other. Using this symbolic connection, they were able to solve an enormous number of important problems in mathematics, physics, and astronomy.

Fourier (1768-1830) studied heat conduction with a series of trigonometric terms to represent functions. Fourier series and integral transforms have applications today in fields as far apart as medicine, linguistics, and music.

Gauss (1777-1855) made the first table of integrals, and with many others continued to apply integrals in the mathematical and physical sciences. Cauchy (1789-1857) took integrals to the complex

Leibniz

Newton

Riemann

domain. Riemann (1826-1866) and Lebesgue (1875-1941) put definite integration on a firm logical foundation.

Liouville (1809-1882) created a framework for constructive integration by finding out when indefinite integrals of elementary functions are again elementary functions. Hermite (1822-1901) found an algorithm for integrating rational functions. In the 1940s Ostrowski extended this algorithm to rational expressions involving the logarithm.

In the 20th century before computers, mathematicians developed the theory of integration and applied it to write tables of integrals and integral transforms. Among these mathematicians were Watson, Titchmarsh, Barnes, Mellin, Meijer, Grobner, Hofreiter, Erdelyi, Lewin, Luke, Magnus, Apelblat, Oberhettinger, Gradshteyn, Ryzhik, Exton, Srivastava, Prudnikov, Brychkov, and Marichev.

In 1969 Risch made the major breakthrough in algorithmic indefinite integration when he published his work on the general theory and practice of integrating elementary functions. His algorithm does not automatically apply to all classes of elementary functions because at the heart of it there is a hard differential equation that needs to be solved. Efforts since then have been directed at handling this equation algorithmically for various sets of elementary functions. These efforts have led to an increasingly complete algorithmization of the Risch scheme. In the 1980s some progress was also made in extending his method to certain classes of special functions.

The capability for definite integration gained substantial power in *Mathematica*, first released in 1988. Comprehensiveness and accuracy have been given strong consideration in the development of *Mathematica* and have been successfully accomplished in its integration code. Besides being able to replicate most of the results from well-known collections of integrals (and to find scores of mistakes and typographical errors in them), *Mathematica* makes it possible to calculate countless new integrals not included in any published handbook.

Introduction

Integral calculus was invented in an attempt to determine the area bounded by the curves. It was done by dividing the region into infinite number of infinitesimal small areas and taking the sum of all these small areas. The term integration signifies summation and is denoted by the symbol v (a destorted form of the letter S, the first letter of the word Sum). The

concept of integration as a sum will be dealt with later on. We shall first discuss integration as a reverse process of differentiation.

Suppose $\frac{d}{dx}$ F(x) = F'(x) = f(x) (say)

then $\int f(x)\,dx = F(x)$.

Thus, we obtain f(x) by differentiating F(x) and get back the original function F(x) by integrating f(x) with respect to x.

For example,

$$\frac{d}{dx}(x^2) = 2x \qquad \therefore \int 2x dx = x^2$$

$$\text{But, } \frac{d}{dx}(x^2+5) = 2x$$

$$\therefore \int 2x\,dx = x^2+5 \text{ also,}$$

Thus, $\int 2x$ dx doesn't give a definite value, and is called indefinite integral. In general, $\int 2x\,dx = x^2+c$, where c is a constant, called constant of integration. This is an arbitrary constant.

$$\text{Similarly, } \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\therefore \int \sec^2 x\ dx = \tan x + c.$$

From these examples, we conclude that the integral of a function is not unique and that if f(x) be any one integral of F(x), then (i) f(x) + c is also its integral, where c is any constant. (ii) every integral of F(x) can be obtained from f(x) + c, by assigning suitable value to c. Thus, $\int F(x)dx$ is really infinite valued. We have discussed this fact in the corollaries of Lagrange's Mean value of theorem. The corollary was that if f'(x) = g'(x) then f(x) and g(x) differ by a constant.

Rules of integration, Properties of integration

Rule I

$$\int kf(x)dx = k\int f(x)dx$$

where k is some constant.

$$\text{e.g.} \qquad \int 5x^2 dx = 5\int x^2 dx$$

Rule II

$$\int [(c_1 f_1(x) + c_2 f_2(x) + \ldots\ldots\ldots + c_n f_n(x)]dx$$

$$= c_1\int f_1(x)dx + c_2\int f_2(x)dx + \ldots\ldots + c_n\int f_n(x)\,dx$$

where $f_1(x), f_2(x), \ldots\ldots\ldots, f_n(x)$ are functions of x.

e.g. $\int(x^3 + 2x + 5)dx = \int x^3dx + \int 2xdx + \int 5dx$.

The rule equally works when the functions are connected by minus sign.

e.g. $\int(x^5 + x^2 + 2)\,dx = \int x^5 dx + \int x^2 dx + \int 2dx$.

Table of Standard Results

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c\ (n \neq -1)$
2. $\int \frac{1}{x} dx = \log x + C$
3. $\int dx = x + C$
4. $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$
5. $\int a^x dx = \frac{a^x}{\log_e a} + C$
6. $\int \sin x\, dx = -\cos x + C$
7. $\int \cos x\, dx = \sin x + C$
8. $\int \sec^2 x\, dx = \tan x + C$
9. $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$
10. $\int \sec x \tan x\, dx = \sec x + C$
11. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$
12. $\int \sinh x\, dx = \cosh x + C$
13. $\int \cosh x\, dx = \sinh x + C$
14. $\int \operatorname{sech}^2 x\, dx = \tanh x + C$
15. $\int \operatorname{cosech}^2 x\, dx = -\coth x + C$
16. $\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$
17. $\int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x + C$
18. $\int \tan x\, dx = \log \sec x + C$
19. $\int \cot x\, dx = \log \sin x + C$
20. $\int \operatorname{cosec} x\, dx = \log\left(\tan\frac{x}{2}\right) = \log(\operatorname{cosec} x - \cot x) + C$
21. $\int \sec x\, dx = \log(\sec x + \tan x) + C$
22. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
23. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$ $(x > a)$
24. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$ $(x < a)$
25. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$
26. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right) = \cosh^{-1}\frac{x}{a} + C$

27. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right) = \sinh^{-1}\frac{x}{a} + C$
28. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$
29. $\int \sqrt{x^2 + a^2}\, dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right) + C$

$$= \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + C$$

30. $\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + C$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + C$$

31. $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$
32. $\int e^{ax}\sin bx\, dx = e^{ax}\left[\frac{a\sin bx - b\cos bx}{a^2 + b^2}\right] + C$
33. $\int e^{ax}\cos bx\, dx = e^{ax}\left[\frac{a\cos bx + b\sin bx}{a^2 + b^2}\right] + C$

Worked out Examples:

1. $\int 2x^6 dx = 2\int x^6 dx = 2\frac{x^{6+1}}{6+1} + C = \frac{2x^7}{7} + C$

2. $\int (8x^3 - 9x^2 + 4x + 6)\, dx = 8\int x^3 dx - 9\int x^2 dx + 4\int x dx + 6\int dx.$

$$= 8\frac{x^4}{4} - 9\frac{x^3}{3} + 4\frac{x^2}{2} + 6x + C$$

$$= 2x^4 - 3x^3 + 2x^2 + 6x + C$$

3. $\int \left[\left(x + \frac{1}{x}\right)\right]^2 dx = \int\left(x^2 + 2x\frac{1}{x} + \frac{1}{x^2}\right) dx = \int x^2 dx + 2\int dx + \int x^{-2} dx$

$$= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} + C = \frac{x^3}{3} + 2x - \frac{1}{x} + C$$

4. $\int\left(\sqrt{x}-\frac{x}{2}+\frac{2}{\sqrt{x}}\right)dx = \int\sqrt{x}\,dx - {}^1/_2\int x dx + 2\int\frac{1}{\sqrt{x}}dx$

$= \int x^{1/2}dx - {}^1/_2\int x dx + 2\int x^{-1/2}dx$

$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^2 + 4x^{1/2} + C$

5. $\int(1-2x)(1+3x)\,dx = \int(1+3x-2x-6x^2)\,dx$

$= \int dx + \int x\,dx - 6\int x^2 dx$

$= x + \frac{x^2}{2} - \frac{6x^3}{3} + C$

$= x + \frac{x^2}{2} - 2x^3 + C$

6. $\int\frac{4x^3-2x}{3x}dx = \int\left(\frac{4x^3}{3x}-\frac{2x}{3x}\right)dx$

$= \frac{4}{3}\int x^2dx - \frac{2}{3}\int dx$

$= \frac{4}{3}\frac{x^3}{3} - \frac{2}{3}x + C$

$= \frac{4}{9}x^3 - \frac{2}{3}x + C.$

7. $\int\frac{a+b\sin x}{\cos^2x}dx = \int\frac{a}{\cos^2x}dx + \int\frac{b\sin x}{\cos^2x}dx$

$= a\int\sec^2x\,dx + b\int\sec x\tan x\,dx$

$= a\tan x + b\sec x + C$

8. $\int\frac{e^{3x}+e^{-3x}}{e^x}dx = \int e^{2x}dx + \int e^{-4x}dx$

$= \frac{e^{2x}}{2} + \frac{e^{-4x}}{-4} + C$

$= \frac{1}{2}e^{2x} - \frac{1}{4}e^{-4x} + C.$

Some Further examples

Example 1.

Integrate $\int\cot^2x\,dx$.

Solution:

Integrate $\int\cot^2x\,dx. = \int(\operatorname{cosec}^2x-1)\,dx = \int\operatorname{cosec}^2x\,dx - \int dx$

$= -\cot x - x + C = -(\cot x + x) + C$

Example 2.

Evaluate $\int\sin x\,\sin 3x\,dx$

Solution:

$I = \frac{1}{2}\int 2\sin x\,\sin 3x\,dx$

$= \frac{1}{2}\int[\cos(x-3x) - \cos(x+3x)]\,dx$

$= \frac{1}{2}\int[\cos 2x - \cos 4x]\,dx = \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 4x}{4}\right] + C$

$= \frac{1}{8}[2\sin 2x - \sin 4x] + C$

Example 3.

Evaluate $\int\frac{\sin x}{1+\sin x}dx$

Solution:

Let $I = \int\frac{\sin x}{1+\sin x}dx$

$I = \int\frac{\sin x}{(1+\sin x)}\times\frac{1-\sin x}{1-\sin x}dx = \int\frac{\sin x-\sin^2x}{1-\sin^2x}dx$

$= \int\left(\frac{\sin x-\sin^2x}{\cos^2x}\right)dx = \int\frac{\sin x}{\cos^2x}dx - \int\frac{\sin^2x}{\cos^2x}dx$

$$= \int \sec x \tan x \, dx - \int \tan^2 x \, dx$$

$$= \int \sec x \tan x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \int \sec x \tan x \, dx - \int \sec^2 x \, dx + \int dx$$

$$= \sec x - \tan x + x + C.$$

Example 4.

$$\int \sqrt{1-\cos x} \, dx$$

Solution:

$$I = \int \sqrt{2 \sin^2 x/2} \, dx = \sqrt{2} \int \sin\frac{x}{2} \, dx$$

$$= -\sqrt{2} \frac{\cos\frac{x}{2}}{\frac{1}{2}} + C = -2\sqrt{2} \cos\frac{x}{2} + C$$

Example 5.

$$\int \frac{dx}{\sin^2 x \cos^2 x}$$

Solution:

Divide N^r and D^r by $\cos^4 x dx$

$$I = \int \frac{\sec^2 x . \sec^2 x}{\tan^2 x} dx = \int \frac{1+t^2}{t^2} dt, \text{ where } \tan x = t$$

$$= \frac{-1}{t} + t + c = -\frac{1}{\tan x} + \tan x + C$$

OR

$$I = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx \qquad (\because \sin^2 x + \cos^2 x = 1)$$

$$= \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx$$

$$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx = \tan x - \cot x + C$$

Example 6.

$$\int \frac{x^2}{x+1} dx$$

Solution: By actual division, we get

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

Now, $$I = \int \left(x - 1 + \frac{1}{x+1} \right) dx$$

$$= \int x dx - \int dx + \int \frac{dx}{x+1}$$

$$= \frac{x^2}{2} - x + \log(x+1) + C$$

EXERCISE 11.1

1. Integrate the Following with respect to x:

a) x^5 b) x^{-6}

c) 0 d) $x^{3/4}$

e) $\frac{1}{\sqrt{x}}$ f) $\frac{1}{\sqrt[3]{x^5}}$

2. a) $2x+3$ b) $3x^2+4x+5$

c) $\frac{a}{x^4} + \frac{b}{x^2}$ (d) $\frac{1+x}{x^3}$

(e) $x(x^2+3)$

3. Evaluate the following

a) $\int \left(6x^2 - \frac{4}{x^2}\right) dx$ (b) $\int (x+1)(2x+3) \, dx$

(c) $\int \frac{ax^3+bx^2+cx+d}{x} dx$

4. Find

(a) $\int \frac{x^3+5x^2-3}{x+2}\,dx$ (b) $\int \left(t+\frac{1}{t}\right)^2 dt$

(c) $\int \left(6\,\text{cosec}^2x-\frac{1}{x^2}\right) dx$ (d) $\int \frac{x^4+x^2+1}{x+1}\,dx$

5. Evaluate:

(a) $\int (\tan^2x-3x^2)\,dx$ (b) $\int \frac{\cos 2x\,dx}{\cos^2x\,\sin^2x}$

(c) $\int \sin^2x.\cos^2x\,dx$

ANSWERS

1. (a) $\frac{x^6}{6}+C$ (b) $-\frac{1}{5x^5}+C$

(c) C (d) $\frac{4}{7}x^{7/4}+C$

(e) $2\sqrt{x}+C$ (f) $\frac{-3}{2x^{2/3}}+C$

2. (a) x^2+3x+C (b) x^3+2x^2+5x+C

(c) $\frac{-a}{3x^3}-\frac{b}{x}+C$ (d) $\frac{-1}{2x^2}-\frac{1}{x}+C$

(e) $\frac{x^4}{4}+\frac{3x^2}{2}+C$

3. (a) $2x^3+\frac{4}{x}+C$ (b) $\frac{2x^3}{3}+\frac{5x^2}{2}+3x+C$

(c) $\frac{ax^3}{3}+\frac{bx^2}{2}+cx+d\log x+k$, when k is a constant of integer

4. (a) $\frac{x^3}{3}+\frac{3x^2}{2}-6x+9\log(x+2)+C$

(b) $\frac{t^3}{3}+2t-\frac{1}{t}+C$ (c) $-6\cot x+\frac{1}{x}+C$

(d) $\frac{x^4}{4}-\frac{x^3}{3}+\frac{x^2}{2}-2x+3\log(x+1)+C$

5. (a) $\tan x-x-x^3+C$ (b) $-\cot x-\tan x+C$

(c) $\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+C$

Method of Substitution

In the process of integration, we depend on the knowledge of the fundamental integrals. But, we can not express all the functions in the form of fundamental integrals. In such cases, special methods of integration are to be followed. One of these methods is method of substitution. This helps us to derive another set of 9 fundamental integrals, which are included in table of standard integral.

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a}+c \text{ if } n\neq -1$$

$$\int \frac{1}{(ax+b)}dx = \frac{\log_e(ax+b)}{a}+c$$

Integrate $\int \frac{1}{\sqrt{2x+1}}dx$

$$\text{Here } I = \int (2x+1)^{-1/2}dx = \frac{(2x+1)^{1/2}}{\frac{1}{2}\times 2}+c$$

$$= \sqrt{2x+1}+c$$

1. In evaluating integral of type $\int \sin^m x\cos^n x dx$, we put cosx = t (new variable) if n is even and we put sinx = t if m is even.

Example: $\int \sin^2 x\cos^3 x\,dx$, put sin x = t

$\therefore$ cosxdx = dt

We write given integral

$$I = \int \sin^2 x\cos^2 x.\cos dx$$

$$= \int \sin^2 x(1-\sin^2 x)\cos x dx$$

Put sinx = t and cosxdx = dt

$$\therefore\ I = \int t^2(1-t^2)dt = \int t^2dt-\int t^4dt = \frac{t^3}{3}-\frac{t^5}{5}+c$$

$$= \frac{1}{3}\sin^3x-\frac{1}{5}\sin^5x+c$$

2. In any integral if a trigonometric ratio contains an angle different form x(say $\sqrt{x}$, or x^2 etc.) it is necessary to put that angle as a new variable 't'. For example (i) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ put $\sqrt{x}$ = t, (ii) $\int e^x.\sec^2(e^x)$ put $e^x = t$

3. In an integral if 'e' has power different from x we have to put that power as a new variable t. For example in $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$, we put $\tan^{-1}x = t$

4. If integral contains the terms like $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$, it would be easy to integrate by letting $x = a\sin\theta$, $x = a\tan\theta$ or $x = a\sec\theta$ respectively.

For the integral $\int \frac{x^2}{\sqrt{a^2-x^2}}$, we put $x = a\sin\theta$ and $dx = a\cos\theta d\theta$

$$\therefore \quad I = \int \frac{a^2\sin^2\theta}{\sqrt{a^2-a^2\sin^2\theta}} \times a\cos\theta d\theta = a^2\int \sin^2\theta d\theta$$

$$= \frac{a^2}{2}\int(1-\cos\theta)d\theta = \frac{a^2}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{a^2}{2}\cdot\frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}} + c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2} + c$$

Example:

Evaluate $\int \frac{x^3}{(x^2-a^2)^{3/2}}dx$

As mentioned earlier put $x = a\sec\theta$ and evaluate. (Left as an exercise)

By adopting rules discussed above one can derive the following standard integrals.

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\{x+\sqrt{x^2-a^2}\} + c$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \log\{x+\sqrt{x^2+a^2}\} + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c$$

Example 1.

$\int(4x+5)^6 dx$

Solution:

Let $I = \int(4x+5)^6 dx$

Put $4x+5 = y$

Differentiating w. r. t. x, we get

$$\frac{d}{dx}(4x+5) = \frac{dy}{dx}$$

$$\text{or, } 4 = \frac{dy}{dx}$$

$$dx = \frac{dy}{4}$$

$$\text{So, } I = \int y^6\frac{dy}{4} = \frac{1}{4}\int y^6 dy = \frac{1}{4}\frac{y^7}{7} + C$$

$$= \frac{1}{28}(4x+5)^7 + C.$$

Example 2

Evaluate: $\int \sin^3x \cos x\, dx$

Solution:

Let $I = \int \sin^3x \cos x\, dx$

Put, $\sin x = y$

Diff. w.r.t. x, we get

$$\frac{d}{dx}(\sin x) = \frac{dy}{dx}$$

$$\text{i.e. } \cos x\, dx = dy$$

$$\text{So, } I = \int y^3 dy$$

$$= \frac{y^4}{4} + C = \frac{\sin^4x}{4} + C$$

Example 3.

Evaluate: $\int 6x\sqrt{3x^2+5}\; dx$

Solution:

Let $I = \int 6x\sqrt{3x^2+5}\ dx$

Put, $3x^2+5 = y$

Diff. w.r.t.x, we get

$$\frac{d}{dx}(3x^2+5) = \frac{dy}{dx}$$

or, $6x\, dx = dy$

So, $I = \int y^{1/2}\, dy = \frac{y^{3/2}}{\frac{3}{2}} + C = \frac{2}{3} y^{3/2} + C$

$$= \frac{2}{3}(3x^2+5)^{3/2} + C$$

Example 4.

Evaluate: $\int x e^{-x^2} dx$

Solution:

Let $I = \int x e^{-x^2} dx$

Put, $x^2 = y$

Diff w.r.t. x, we get

$$\frac{d}{dx}(x^2) = \frac{dy}{dx}$$

$2x\, dx = dy$

$\therefore x dx = \frac{dy}{2}$

So, $I = \int e^{-x^2}(x\, dx) = \int e^{-y}\frac{dy}{2} = \frac{1}{2}\int e^{-y} dy$

$$= \frac{1}{2}\frac{e^{-y}}{-1} + C = -\frac{1}{2}e^{-x^2} + C.$$

Some special forms of method of substitution:

Case I

Integral of type $\int [f(x)]^n\ f'(x)\, dx$

Here, $I = \int [f(x)]^n)\ f'(x)\, dx$

Put, $f(x) = y$

Differentiating w.r.t.x, we get

$$\frac{d}{dx}[f(x)] = \frac{dy}{dx}$$

$\therefore f'(x)\, dx = dy$

So, $I = \int y^n\, dy = \frac{y^{n+1}}{n+1} + C$

$\therefore \quad I = \frac{[f(x)]^{n+1}}{n+1} + C$

Case II

Integral of type $\int \frac{f'(x)}{f(x)} dx.$

Here, $I = \int \frac{f'(x)}{f(x)} dx$

Put, $f(x) = y$

Differentiating w.r.t.x, we get

$f'(x)\, dx = dy$

$\therefore I = \int \frac{dy}{y} = \log y + C$

$\therefore \quad \int \frac{f'(x)}{f(x)} dx = \log f(x) + C$

Example 1.

Evaluate : $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} dx$

Solution:

Let $I = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} dx$

Put, $\sin^{-1}x = y$

Differentiating w.r.t x, we get

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$$\frac{d}{dx}(\sin^{-1}x) = \frac{dy}{dx}$$

$$\text{or, } \frac{1}{\sqrt{1-x^2}}\,dx = dy$$

So, $$I = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx = \int e^y\,dy = e^y + C$$

i.e. $$I = e^{\sin^{-1}x} + C$$

Example 2.

Evaluate: $\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\,dx$

Solution:

Let $I = \int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\,dx$

Put, $e^{2x} - e^{-2x} = y$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(e^{2x} - e^{-2x}) = \frac{dy}{dx}$$

$$\text{or, } 2e^{2x} + 2e^{-2x} = \frac{dy}{dx}$$

$$\text{or, } (e^{2x} + e^{-2x})\,dx = \frac{dy}{2}$$

So, $$I = \int \frac{1}{y}\frac{dy}{2}$$

$$= \frac{1}{2}\int \frac{dy}{y} = \frac{1}{2}\log y + C$$

i.e. $$I = \frac{1}{2}\log(e^{2x} - e^{-2x}) + C.$$

Example 3.

Evaluate: $\int \frac{e^x(1+x)}{\cos^2(xe^x)}\,dx$

Solution:

Let $I = \int \frac{e^x(1+x)}{\cos^2(xe^x)}\,dx$. As mentioned earlier we put, $xe^x = y$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(xe^x) = \frac{dy}{dx}$$

$$\text{or, } xe^x + e^x = \frac{dy}{dx}$$

$$e^x(x+1)\,dx = dy$$

So, $$I = \int \frac{dy}{\cos^2 y} = \int \sec^2 y\,dy = \tan y + C$$

$$\therefore \quad I = \tan(xe^x) + C$$

Example 4.

Evaluate: $\int \frac{dx}{(x+\sqrt{x})}\,dx.$

Solution:

Let $I = \int \frac{dx}{(x+\sqrt{x})}\,dx.$

Put, $\sqrt{x} = y$ i.e. $x = y^2$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$$

$$\text{or, } 1 = \frac{d(y^2)}{dy}\frac{dy}{dx}$$

$$\text{or, } 1 = 2y\frac{dy}{dx}$$

$$\therefore dx = 2y\,dy$$

$$\text{So, } I = \int \frac{2y\,dy}{y^2+y} = 2\int \frac{y\,dy}{y(y+1)} = 2\int \frac{dy}{y+1}$$

$$= 2\log(y+1)$$

$$\therefore I = 2\log(\sqrt{x}+1) + c$$

Example 5.

Integrate $\frac{\sin x \cos x}{a^2\sin^2x+b^2\cos^2x}$ w.r.t. x.

Solution:

Let $I = \int \frac{\sin x \cos x \, dx}{a^2 \sin^2 x+b^2 \cos^2 x}$

Put, $a^2\sin^2x + b^2\cos^2x = y$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(a^2\sin^2x + b^2\cos^2x) = \frac{dy}{dx}$$

$$\text{or, } (a^2 2\sin x \cos x - b^2 \, 2\cos x \sin x) = \frac{dy}{dx}$$

$$\text{or, } 2\sin x \cos x (a^2 - b^2) = \frac{dy}{dx}$$

$$\therefore \sin x \cos x \, dx = \frac{dy}{2(a^2 - b^2)}$$

$$\text{Thus, } I = \int \frac{\sin x \cos x \, dx}{a^2\sin^2x+b^2\cos^2x} = \int \frac{dy}{2(a^2-b^2)y}$$

$$= \frac{1}{2(a^2-b^2)}\int \frac{dy}{y} = \frac{1}{2(a^2-b^2)} \log y + C$$

$$\therefore I = \frac{1}{2(a^2-b^2)} \log(a^2\sin^2x+b^2\cos^2x)+C$$

Example 6.

Evaluate: $\int \frac{5}{x[1+(\log x)]^m} dx$

Solution:

Let $I = \int \frac{5}{x[1+(\log x)]^m} dx$

Put, $1 + \log x = y$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(1+\log x) = \frac{dy}{dx}$$

$$\text{or, } \frac{1}{x} dx = dy$$

$$\text{So, } I = 5\int \frac{1}{x} dx \frac{1}{[1+(\log x)]^m} = 5\int \frac{1}{y^m} dy$$

$$= 5\int y^{-m} dy = 5\frac{y^{-m+1}}{-m+1} + C$$

$$= \frac{5}{1-m}[1+(\log x)]^{1-m} + C.$$

Example 7.

Evaluate: $\int \frac{1+\sin 2x}{x+\sin^2 x} dx$

Solution:

Let $I = \int \frac{1+\sin 2x}{x+\sin^2 x} dx$

$$\text{So, } I = \int \frac{1+2\sin x \cos x}{x+\sin^2 x} dx = \int \frac{\frac{d}{dx}(x+\sin^2 x)}{x+\sin^2 x} dx$$

$$= \log(x+\sin^2 x) + C \qquad \left(\because \int \frac{f'(x)}{f(x)} dx = \log[f(x)]+c\right)$$

Example 8.

Evaluate: $\int \frac{1}{e^x+1} dx$

Solution:

$$\text{Let } I = \int \frac{1}{e^x+1} dx = \int \frac{1}{(e^x+1)} dx = \int \frac{1}{\left(\frac{1}{e^{-x}}+1\right)} dx = \int \frac{e^{-x}}{1+e^{-x}} dx$$

Put, $1+e^{-x} = y$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(1+e^{-x}) = \frac{dy}{dx} \qquad \text{or, } -e^{-x} = \frac{dy}{dx}$$

$$\therefore e^{-x} dx = -dy$$

$$I = \int \frac{-dy}{y} = -\log y + C$$

$$\text{So, } I = -\log(1+e^{-x}) + C$$

EXERCISE 11.2

Evaluate the following integrals:

1. $\int \frac{16x\,dx}{8x^2+2}$

2. $\int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$

3. $\int x e^{x^2}\,dx$

4. $\int \frac{x\,dx}{\sqrt{8x^2+1}}$

5. $\int \frac{\sin x}{3+4\cos x}\,dx$

6. $\int e^x \sec^2(e^x)\,dx$

7. $\int \frac{dx}{\sqrt{x}(1+x)}$

8. $\int e^x\sqrt{3+4e^x}\,dx$

9. $\int \frac{dx}{x-\sqrt{x}}$

10. $\int \frac{dx}{x\log x}$

11. $\int \frac{e^x}{1+e^x}\,dx$

12. $\int e^{3x}\,dx$

13. $\int \frac{\tan\sqrt{x}}{\sqrt{x}}\,dx$

14. $\int \frac{x\,dx}{\sqrt{1-4x^2}}$

15. $\int \frac{\sin^2 2x}{1+\cos 2x}\,dx$

16. $\int \frac{dy}{y\sqrt{y^2-1}}$

17. $\int \frac{x\,dx}{(3x^2+4)^3}$

18. $\int \frac{x^2\,dx}{\sqrt{x^3+5}}$

19. $\int 3^{4x}\,dx$

20. $\int \cos^2 x\sin x\,dx$

21. $\int \cot^3 x\,\mathrm{cosec}^2 x\,dx$

22. $\int \frac{dx}{\sqrt{e^{2x}-1}}$ (Hint: Multiply numerator and denominator by e^x)

23. $\int \frac{x^8}{(1-x^3)^{1/3}}\,dx$ (Hint: put $1-x^3 = y^3$)

24. $\int e^{\tan 3x}\sec^2 3x\,dx$

25. $\int x^{1/3}\sqrt{x^{4/3}-1}\,dx$

26. $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx$

27. $\int \frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta$

28. $\int \frac{\sqrt{\tan^{-1}x}}{2(1+x^2)}\,dx$

29. $\int \sin x\sqrt{1-\cos 2x}\,dx$

30. $\int x\sin^3(x^2)\cos(x^2)\,dx$

31. $\int \sin^3 x\cos^4 x\,dx$

32. $\int \frac{\cos(\log x)}{x}\,dx$

33. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\,dx$

34. $\int \frac{3e^{2x}+3e^{4x}}{e^x+e^{-x}}\,dx$

35. $\int \log(x^2+1)\;2x\left((1+x^2)\right)^{-1}\,dx$

36. $\int \frac{\sin 2x}{(\sin^4 x+\cos^4 x)}\,dx$ [Hint: Divide numerator and denominator by $\cos^4 x$ and Put $\tan^2 x = y$]

37. $\int \frac{1}{(3\tan x+1)\cos^2 x}\,dx$

38. $\int \frac{1}{x\log x\,\log(\log x)}\,dx$

39. $\int \sec^4 x\,dx$

40. $\int \tan^5 x\,dx$

ANSWERS

(1) $\log(8x^2+2)+C$

(2) $2\sqrt{1+\sin x}+C$

(3) $\frac{1}{2}e^{x^2}+C$

(4) $\frac{1}{8}\sqrt{8x^2+1}+C$

(5) $\frac{-1}{4}\log(3+4\sin x)+C$

(6) $\tan(e^x)+C$

(7) $2\tan^{-1}(\sqrt{x})+C$

(8) $\frac{1}{6}(3+4e^x)^{3/2}+C$

(9) $2\log(\sqrt{x}-1)+C$

(10) $\log(\log x)+C$

(11) $\log(1+e^x)+C$

(12) $\frac{1}{3}e^{3x}+C$

(13) $\log(\sec^2\sqrt{x})+C$

14. $\frac{-1}{4}\sqrt{1-4x^2}+C$

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(15) $x - \frac{\sin 2x}{2} + C$

(16) $\sec^{-1}(y) + C$

(17) $\frac{-1}{12(3x^2+4)^2} + C$

(18) $\frac{2}{3}\sqrt{x^3+5} + C$

(19) $\frac{3^{4x}}{4\log 3} + C$

(20) $\frac{-1}{3}\cos^3 x + C$

(21) $\frac{-1}{4}\cot^4 x + C$

(22) $\sec^{-1}(e^x) + C$

(23) $\frac{-1}{2}(1-x^3)^{2/3} + \frac{2}{5}(1-x^3)^{5/3} - \frac{1}{8}(1-x^3)^{8/3} + C$

(24) $\frac{1}{3}e^{\tan 3x} + C$

(25) $\frac{1}{2}\left(x^{4/3}-1\right)^{3/2} + C$

(26) $2e^{\sqrt{x+1}} + C$

(27) $-2\sqrt{(1+\cos\theta)} + C$

(28) $\frac{1}{3}(\tan^{-1}x)^{3/2} + C$

(29) $\frac{x}{\sqrt{2}} - \frac{\sin 2x}{2\sqrt{2}} + C$

(30) $\frac{1}{8}\sin^4(x^2) + C$

(31) $-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

(32) $\sin(\log x) + C$

(33) $\frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C$

(34) $e^{3x} + C$

(35) $\frac{\log^2(x^2+1)}{2} + C$

(36) $\tan^{-1}(\tan^2 x) + C$

(37) $\frac{1}{3}\log(3\tan x + 1) + C$

(38) $\log[\log(\log x)] + C$

(39) $\tan x + \frac{\tan^3 x}{3} + C$

(40) $\frac{1}{4}\sec^4 x - \sec^2 x + \log(\sec x) + C$

Integrals of Trigonometric Functions:

In the standard forms, we have already seen that

a. $\int \sin x\, dx = -\cos x$

b. $\int \cos x\, dx = \sin x.$

Now, we derive the following formulae:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{\frac{d}{dx}(\cos x)}{(\cos x)} = -\log(\cos x) + C$$

$$= \log(\cos x)^{-1} + C$$

$$= \log(\sec x) + C$$

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \int \frac{\frac{d}{dx}(\sin x)}{\sin x}\, dx$$

$$= \log(\sin x) + C$$

$$\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)}\, dx$$

$$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x}\, dx$$

$$= \int \frac{\frac{d}{dx}(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)}$$

$$= \log(\operatorname{cosec} x - \cot x) + C$$

which can also be put in the form

$$\log\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) + C = \log\left(\frac{1-\cos x}{\sin x}\right) + C$$

$$= \log\left(\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) + C = \log\left(\tan\frac{x}{2}\right) + C$$

$$\therefore \int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x) + C = \log\left(\tan\frac{x}{2}\right) + C$$

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx = \int \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \log(\sec x + \tan x)$$

which can be put in the form

$\log\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)$

$$= \log\left(\frac{1+\sin x}{\cos x}\right) = \log\left(\frac{\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right)$$

$$= \log\left\{\frac{(\cos\frac{x}{2}+\sin\frac{x}{2})^2}{(\cos\frac{x}{2}-\sin\frac{x}{2})(\cos\frac{x}{2}+\sin\frac{x}{2})}\right\} = \log\left(\frac{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right)$$

$$= \log\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) = \log\left(\frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}\right) = \log\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$$

$$\therefore \int \sec x\,dx = \log(\sec x+\tan x) = \log\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$$

Thus, we have derived

1. $\int \tan x\,dx = \log(\sec x) + C$
2. $\int \cot x\,dx = \log(\sin x) + C$
3. $\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x) + C = \log\left[\tan\left(\frac{x}{2}\right)\right] + C$
4. $\int \sec x\,dx = \log(\sec x + \tan x) + C = \log\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] + C$

Example 1.

Evaluate: $\int \frac{\cos 2x}{\cos x}\,dx$

Solution:

$$I = \int \frac{\cos 2x}{\cos x}\,dx = \int\left(\frac{2\cos^2 x-1}{\cos x}\right)dx$$

$$= \int\left(\frac{2\cos^2 x}{\cos x}-\frac{1}{\cos x}\right)dx = 2\int \cos x\,dx - \int \sec x\,dx$$

$$= 2\sin x - \log(\sec x+\tan x) + C.$$

Some Standard Integrals

Example 1.

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right)+C$$

Do yourself by putting

$x = a\tan\theta$

Example 2.

$$\int \frac{dx}{x^2-a^2}$$

Solution:

$$I = \int \frac{dx}{x^2-a^2}$$

Now, $\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$

$$\therefore I = \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx = \frac{1}{2a}\left[\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right]$$

$$= \frac{1}{2a}\left[\log(x-a)-\log(x+a)\right]+C = \frac{1}{2a}\log\frac{x-a}{x+a}+C$$

$$\therefore \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+C$$

Example 3.

$$\int\frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+C$$

Do yourself, taking $\frac{1}{a^2-x^2} = \frac{1}{2a}\left(\frac{1}{a+x}+\frac{1}{a-x}\right)$.

Example 4.

$$\int \frac{dx}{x\sqrt{x^2 - a^2}}$$

Solution:

$$I = \int \frac{dx}{x\sqrt{x^2 - a^2}}$$

Put $x = a\sec\theta$

$\therefore dx = a\sec\theta \tan\theta\, d\theta$

$$\text{So, } I = \int \frac{a\sec\theta \tan\theta\, d\theta}{a\sec\theta\ a\tan\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$$

But $x = a\sec\theta \Rightarrow \sec\theta = \frac{x}{a}$

$$\therefore \theta = \sec^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

Example 5.

Evaluate $\int \frac{dx}{\sqrt{x^2-2x+5}}$

Solution:

$$I = \int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x)^2-2 \times 1+(1)^2+(2)^2}}$$

$$= \int \frac{dx}{\sqrt{(x-1)^2+(2)^2}}$$

Put, $x - 1 = y$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(x-1) = \frac{dy}{dx} \qquad \therefore dx = dy$$

$$\text{So, } I = \int \frac{dy}{\sqrt{(y)^2+(2)^2}} = \log\left(y + \sqrt{y^2 + 2^2}\right) + C$$

$$\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log\{x + \sqrt{x^2+a^2}\}$$

Putting, the value of y, we get,

$$\log\left(x - 1 + \sqrt{(x-1)^2 + (2)^2}\right) + C = \log\left(x - 1 + \sqrt{x^2 - 2x + 5}\right) + C$$

Example 6.

Evaluate: $\int \frac{dx}{\sqrt{2x-x^2}}$

Solution:

$$I = \int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(1-2x+x^2)}} = \int \frac{dx}{\sqrt{(1)^2-(x-1)^2}}$$

Put $x - 1 = y$

Differentiating w.r.t. x, we get

$dx = dy$

$$\therefore I = \int \frac{dy}{\sqrt{(1)^2-(y)^2}} = \sin^{-1}\left(\frac{y}{1}\right) + C$$

Putting the value of y, $I = \sin^{-1}(x-1) + C$.

Note: This problem can be solved by putting $x = t^2$.

Example 7.

Evaluate: $\int \frac{3x+7}{(2x^2+3x-2)}\, dx.$

Solution:

$$\text{Let } (3x+7) = A\frac{d}{dx}(2x^2+3x-2) + B$$

$$\therefore (3x+7) = A(4x+3) + B.$$

$$\text{i.e } 3x+7 = 4Ax + (3A+B)$$

Equating the coefficients of x and constant terms

$$4A = 3 \text{ and } 3A+B = 7$$

$$A = \frac{3}{4} \text{ and } 3 \times \frac{3}{4} + B = 7$$

$$A = \frac{3}{4} \text{ and } B = 7 - \frac{9}{4} = \frac{19}{4}$$

$$\text{So, } I = \int \frac{\frac{3}{4}\frac{d}{dx}(2x^2+3x-2) + \frac{19}{4}}{(2x^2+3x-2)}\, dx$$

$$= \frac{3}{4}\int \frac{\frac{d}{dx}(2x^2+3x-2)}{(2x^2+3x-2)} + \frac{19}{4}\int \frac{1}{2(x^2+{}^3/_2\, x-1)}\, dx$$

$$= \frac{3}{4}\log(2x^2+3x-2) + \frac{19}{4\times 2}\int \frac{dx}{(x)^2+2\,x\,\frac{3}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2-1}$$

$$= \frac{3}{4}\log(2x^2+3x-2)+\frac{19}{8}\int\frac{dx}{\left(x+\frac{3}{4}\right)^2-\frac{9}{16}-1}$$

$$= \frac{3}{4}\log(2x^2+3x-2)+\frac{19}{8}\int\frac{dx}{\left(x+\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}$$

$$= \frac{3}{4}\log(2x^2+3x-2)+\frac{19}{8}\,\frac{1}{2\,\frac{5}{4}}\log\left(\frac{x+\frac{3}{4}-\frac{5}{4}}{x+\frac{3}{4}+\frac{5}{4}}\right)+C$$

$$= \frac{3}{4}\log(2x^2+3x-2)+\frac{19}{20}\log\left[\frac{2x-1}{2(x+2)}\right]+C.$$

Example 8.

Evaluate: $\int\frac{dx}{4+5\sin^2x}$.

Solution:

$$I=\int\frac{dx}{4+5\sin^2x}=\int\frac{\frac{dx}{\cos^2x}}{\frac{4+5\sin^2x}{\cos^2x}} \quad \text{(Dividing by } \cos^2x \text{ in N \& D)}$$

$$=\int\frac{\sec^2x\,dx}{4\sec^2x+5\tan^2x}=\int\frac{\sec^2x\,dx}{4(1+\tan^2x)+5\tan^2x}=\int\frac{\sec^2x\,dx}{4+9\tan^2x}$$

Put, 3tan x = y

Differentiating we get,

$3\sec^2x\,dx = y$

i.e. $\sec^2x\,dx = \frac{dy}{3}$

So, $I = \frac{1}{3}\int\frac{dy}{(2)^2+(y)^2} = \frac{1}{3}\,\frac{1}{2}\,\tan^{-1}\frac{y}{2}+C$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right)+C$$

Example 9.

Evaluate $\int\frac{dx}{\sqrt{(x-a)(x-b)}}$

Solution:

$$I = \int\frac{dx}{\sqrt{x^2-(a+b)\,x+ab}} = \int\frac{dx}{\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}}$$

$$= \log\left[\left(x-\frac{a+b}{2}\right)+\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{ab}{2}\right)^2}\right]+C$$

$$= \log\left[\frac{2x-a-b}{2}+\sqrt{(x-a)(x-b)}\right]+C$$

$$= \log\left[\frac{(x-a)+(x-b)}{2}+\sqrt{(x-a)(x-b)}\right]+C$$

$$= \log\left[\left(\sqrt{x-a}\right)^2+2\sqrt{(x-a)(x-b)}+\left(\sqrt{x-b}\right)^2\right]+C$$

$$= \log\left[\sqrt{x-a}+\sqrt{x-b}\right]^2-\log 2+C$$

$$= \log\left(\sqrt{x-a}+\sqrt{x-b}\right)^2+C_1 \quad (\because C-\log 2 \text{ is an other constant})$$

$$= 2\log\left(\sqrt{x-a}+\sqrt{x-b}\right)+C_1$$

Note: This sum can also be done by putting $x-a = t^2$

Thus, $x-b = t^2+a-b$ and $dx = 2t\,dt$ (Exercise for student)

Example 10.

Find $\int\frac{\sqrt{x}}{1+x}\,dx$

Solution:

$$I = \int\frac{\sqrt{x}}{1+x}\,dx$$

Put, $x = y^2$

$\therefore dx = 2y\,dy$

Now,

$$I = \int \frac{y}{1+y^2}\, 2y\, dy = 2\int \frac{y^2}{1+y^2}\, dy$$

$$= 2\int \frac{(1+y^2)-1}{1+y^2}\, dy$$

$$= 2\left[\int \frac{1+y^2}{1+y^2}\, dy - \int \frac{1}{1+y^2}\, dy\right]$$

$$= 2\left[\int dy - \int \frac{dy}{(y)^2+(1)^2}\right] = 2\left[y - \tan^{-1}\left(\frac{y}{1}\right)\right] + C$$

Putting the value of y, we get

$$I = 2\left[\sqrt{x} - \tan^{-1}\left(\sqrt{x}\right)\right] + C$$

Example 11.

Evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$ compare with problem given below.

Solution:

$$I = \int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{x^2+2x+1-1}}$$

$$= \int \frac{dx}{(x+1)\sqrt{(x+1)^2-(1)^2}}$$

Put, $x+1 = y$

Diff. w. r. t. x, we get

$dx = dy$

$$\therefore I = \int \frac{dy}{y\sqrt{y^2-1}}$$

$$= \sec^{-1}(y) + C \qquad \left(\because \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C\right)$$

Putting the value of y

$$I = \sec^{-1}(x+1) + C.$$

Note: This type of sum can be solved by putting linear factor i.e. $x+1 = \frac{1}{y}$, then the integral reduces to standard form.

Form $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$

Example 12.

$$\int \frac{dx}{(2x+1)\sqrt{x^2+2x+2}}$$

Solution:

Let $I = \int \frac{dx}{(2x+1)\sqrt{x^2+2x+2}}$

Put, $2x+1 = \frac{1}{y}$

$$\therefore \frac{d}{dx}(2x+1) = \frac{d}{dy}(y^{-1})\frac{dy}{dx}$$

$$\text{or, } 2dx = -y^{-2}dy$$

$$\text{or, } dx = \frac{-1}{2y^2}\, dy$$

So,

$$I = \int\left(\frac{\frac{1}{2y^2}dy}{\frac{1}{y}\sqrt{\left(\frac{1-y}{2y}\right)^2 + 2\left(\frac{1-y}{2y}\right) + 2}}\right) = \int\left(\frac{\frac{1}{2y^2}dy}{\frac{1}{y}\sqrt{\frac{(1-y)^2+4y(1-y)+2.4y^2}{4y^2}}}\right)$$

$$= \int \frac{dy}{\sqrt{1-2y+y^2+4y-4y^2+8y^2}} = \int \frac{dy}{\sqrt{5y^2+2y+1}}$$

$$= \int \frac{dy}{\sqrt{5(y^2+\frac{2}{5}y+\frac{1}{5})}} = -\frac{1}{\sqrt{5}}\int \frac{dy}{\sqrt{(y)^2+2y\frac{1}{5}+(\frac{1}{5})^2+\frac{1}{5}-\frac{1}{25}}}$$

$$= -\frac{1}{\sqrt{5}}\int \frac{dy}{\sqrt{(y+\frac{1}{5})^2+(\frac{2}{5})^2}}$$

$$= -\frac{1}{\sqrt{5}}\log\left[\left(y+\frac{1}{5}\right) + \sqrt{\left(y+\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2}\right]$$

$$= -\frac{1}{\sqrt{5}}\log\left[\frac{5y+1+\sqrt{(5y+1)^2+(2)^2}}{5}\right] + C$$

$$= -\frac{1}{\sqrt{5}}\log[5y+1+\sqrt{25y^2+10y+5}\,]+C$$

$$= -\frac{1}{\sqrt{5}}\log\left[5\left(\frac{1}{2x+1}\right)+1+\sqrt{25\left(\frac{1}{2x+1}\right)^2+10\left(\frac{1}{2x+1}\right)+5}\,\right]+C$$

$$= -\frac{1}{\sqrt{5}}\log\left[\frac{2x+6+\sqrt{20x^2+40x+40}}{2x+1}\right]+C$$

Integrals Reducible to Standard Form

When, we have to integrate the functions of the form $\frac{1}{a+b\cos x}$, $\frac{1}{a+b\sin x}$ and $\frac{1}{a+b\cos x+c\sin x}$ then the most general way to find the solution is to substitute $\tan\frac{x}{2} = y$. By substituting this, the above forms of integrals will be reduced to standard algebraic forms which we can evaluate easily.

We follow the following results:

Put, $\tan\frac{x}{2} = y$..................(i)

Then, from trigonometry

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2y}{1+y^2}$$

$$\text{i.e. } \sin x = \frac{2y}{1+y^2}$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-y^2}{1+y^2}$$

Also, differentiating (i) w.r.t. x, we get

$$\sec^2\frac{x}{2}\,\frac{1}{2}\,dx = dy$$

$$\text{i.e. } dx = \frac{2\,dy}{1+y^2}$$

This process helps us to reduce the given integral into standard form. The following examples help us to understand it.

Example 13.

Evaluate: $\int\frac{dx}{4+5\sin x}$

Solution:

Let $I = \int\frac{dx}{4+5\sin x}$

Put, $\tan^{x}/_{2} = y$

Then, $\sin x = \frac{2y}{1+y^2}$ and $\cos x = \frac{1-y^2}{1+y^2}$

and $dx = \frac{2\,dy}{1+y^2}$

$$\text{So, } I = \int\frac{2\,dy/1+y^2}{4+5\frac{2y}{1+y^2}} = \int\frac{2\,dy}{4y^2+10y+4} = \frac{2}{4}\int\frac{dy}{\left(y^2+\frac{10}{4}y+1\right)}$$

$$= \frac{1}{2}\int\frac{dy}{(y)^2+2\,y\frac{5}{4}+\left(\frac{5}{4}\right)^2+1-\left(\frac{5}{4}\right)^2} = \frac{1}{2}\int\frac{dy}{\left(y+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\,\frac{1}{\frac{2\times3}{4}}\log\left(\frac{y+\frac{5}{4}-\frac{3}{4}}{y+\frac{5}{4}+\frac{3}{4}}\right)+C$$

$$= \frac{1}{3}\log\left(\frac{y+\frac{1}{2}}{y+2}\right)+C$$

Substituting the value of y, we get

$$I = \frac{1}{3}\log\left(\frac{\tan\frac{x}{2}+\frac{1}{2}}{\tan\frac{x}{2}+2}\right)+C = \frac{1}{3}\log\left\{\frac{2\tan\frac{x}{2}+1}{2(\tan\frac{x}{2}+2)}\right\}+C$$

Example 14.

Evaluate: $\int\frac{dx}{2+\cos x+\sin x}$

Solution:

Let $I = \int \frac{dx}{2+\cos x+\sin x}$

Put $\tan x/2 = y$

Then, $\sin x = \frac{2y}{1+y^2}$ and $\cos x = \frac{1-y^2}{1+y^2}$ and $dx = \frac{2\,dy}{1+y^2}$

$$\text{So, } I = \int \frac{\frac{2dy}{1+y^2}}{2+\frac{1-y^2}{1+y^2}+\frac{2y}{1+y^2}} = 2\int \frac{dy}{y^2+2y+3}$$

$$= 2\int \frac{dy}{(y)^2+2\,y\,1+(1)^2+\sqrt{(2)^2}}$$

$$= 2\int \frac{dy}{(y+1)^2+\sqrt{(2)}^2} = 2\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y+1}{\sqrt{2}}\right)+C$$

$$\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\right]$$

$$= \sqrt{2}\tan^{-1}\left(\frac{1+\tan x/2}{\sqrt{2}}\right)+C \qquad (\because y = \tan x/2)$$

Example 15.

Form $\int \frac{dx}{a\sin x+b\cos x} = I$ (say)

Put $a = r\cos\theta$ and $b = r\sin\theta$

$\therefore a\sin x + b\cos x = r\sin(x+\theta)$

And $r^2 = a^2+b^2$ and $\tan\theta = b/a$

So, $I = \int \frac{dx}{r\sin(x+\theta)} = 1/r \int \text{cosec}(x+\theta)\,dx$

Put $x+\theta = y$ then $dx = dy$

$$I = \frac{1}{r}\int \text{cosec}\,y\,dy = \frac{1}{r}\log\left(\tan\frac{y}{2}\right)+C$$

$$\therefore \quad = \frac{1}{\sqrt{a^2+b^2}}\log\left(\tan\frac{x+\theta}{2}\right)+C$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\left(\tan\frac{x+\tan^{-1}(b/a)}{2}\right)+C$$

Example 16. Evaluate: $\int \frac{dx}{5+4\cos x}$ 2009

Solution:

We have $I = \int \frac{dx}{5+4\cos x}$

$$= \int \frac{dx}{5(\cos^2\frac{x}{2}+\sin^2\frac{x}{2})+4(\cos^2\frac{x}{2}-\sin^2\frac{x}{2})}$$

$$I = \int \frac{dx}{9\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}$$

Now divide N^r and D^r by $\cos^2\frac{x}{2}$, we get,

$$I = \int \frac{\sec^2\frac{x}{2}}{9\tan^2\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = t$

$\therefore \quad \text{sex}^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$

$\sec^2\frac{x}{2}dx = 2dt$

Then; $I = \int \frac{2dt}{9+t^2} = \frac{2}{3}\tan^{-1}(t/3)+c$

$$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\cdot\tan\frac{x}{2}\right)+c$$

- For integral of type $\int \frac{1}{a+b\cos^2 x}dx$.

- But for $\int \frac{dx}{4\sin^2 x + 5\cos^2 x}$ in which denominator already contains $\cos^2 x$, We just divided Numerator and Denominator by $\cos^2 x$, then

$$I = \int \frac{\sec^2 x}{4\tan^2 x + 5} dx$$

Put $\tan x = t$

$\sec^2 x dx = dt$

$\therefore$

Then,

$$I = \int \frac{dt}{4t^2+5} = \frac{1}{4}\int \frac{dx}{t^2 + (\frac{\sqrt{5}}{2})^2}$$

$$= \frac{1}{4} \times \frac{1}{\frac{\sqrt{5}}{2}} \tan^{-1}\left[\frac{t}{(\frac{\sqrt{5}}{2})}\right] = \frac{1}{2}\sqrt{5} \tan^{-1} \frac{2t}{\sqrt{5}} + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left[\frac{2}{\sqrt{5}} \tan x\right] + c$$

- Integral of the form $\int \frac{1}{\sqrt{(x-a)(x-b)}} dx$ or $\int \frac{dx}{\sqrt{(a-x)(x-b)}}$ can be evaluated by letting one of the factor of denominator as t^2.

For example, $\int \frac{dx}{\sqrt{(x-1)(x+2)}}$, we put $x - 1 = t^2$

$x = t^2 + 1 \quad dx = 2tdt$

$$\text{Then, } I = \int \frac{2tdt}{\sqrt{t^2(t^2+1+2)}} = 2\int \frac{dt}{\sqrt{t^2+3}}$$

$$= 2\log(t + \sqrt{t^2+3}) + c$$

$$= 2\log[\sqrt{x-1} + \sqrt{x-1+3}\,] + c$$

$$= 2\log(\sqrt{x-1} + \sqrt{x+2}) + c$$

- For integral of the type $\int \sqrt{\frac{ax+b}{cx+d}} dx$, we put $cx + d = t^2$

For example in evaluating $\int \sqrt{\frac{5-x}{x-2}} dx$, we put $x - 2 = t^2$,

i.e. $x = t^2 + 2$ so that $dx = 2tdt$.

The given integral

$$I = \int \sqrt{\frac{5-t^2-2}{t^2}} \times 2tdt$$

$$I = \int \sqrt{3-t^2}\, dt, \text{ using standard integral}$$

$$= \frac{t\sqrt{3-t^2}}{2} + \frac{3}{2}\sin^{-1}\frac{t}{\sqrt{3}} + c$$

$$= \frac{\sqrt{x-2}.\sqrt{3-x+2}}{2} + \frac{3}{2}\sin^{-1}\left(\sqrt{\frac{x-2}{3}}\right) + c$$

$$= \frac{1}{2}(\sqrt{x-2})(\sqrt{5-x}) + \frac{3}{2}\sin^{-1}\sqrt{\frac{x-2}{3}} + c$$

- The integral is of the form $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$ can be evaluated by putting $px + q = \frac{1}{t}$. For example for $\int \frac{dx}{(x+1)\sqrt{x^2-1}}$

We, put $x + 1 = \frac{1}{t}$

$$dx = -\frac{1}{t^2} dt$$

$\therefore$

$$x = \frac{1}{t} - 1$$

$$x = \frac{1-t}{t}$$

$$x^2 = \frac{1-2t+t^2}{t^2}$$

$$\text{Then, } I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1-2t+t^2}{t^2}-1}} = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}.\frac{1}{t}\sqrt{1-2t}} dt$$

$$= -\int (1-2t)^{-1/2} dt = \frac{(\sqrt{1-2t})\times 2}{-2} + c$$

$$= \sqrt{1-2t} + c = \sqrt{1-\frac{2}{x+1}} + c = \sqrt{\frac{x-1}{x+1}} + c$$

EXERCISE 11.3

A) Evaluate the integrals

1. $\int \frac{dx}{\sqrt{x^2-2x+5}}$

2. $\int \frac{x\,dx}{x^2-2x+5}$

3. $\int \frac{x\,dx}{\sqrt{x^2-2x+5}}$

4. $\int \frac{x+1}{3+2x-x^2}\,dx$

5. $\int \frac{3\,dx}{\sqrt{15-6x-x^2}}$

6. $\int \frac{a-x}{\sqrt{2ax-x^2}}\,dx$

7. $\int \frac{dx}{\sqrt{x^2+2x}}$

8. $\int \frac{(1-x)dx}{\sqrt{8+2x-x^2}}$

9. $\int \frac{x\,dx}{\sqrt{x^2+4x+5}}$

10. $\int \frac{dx}{(x+3)\sqrt{x^2+6x+10}}$

11. $\int \frac{(2x+3)dx}{4x^2+4x+5}$

12. $\int \frac{x\,dx}{\sqrt{x^2+4x+13}}$

13. $\int \frac{2x^2+3x+4}{x^2+6x+10}\,dx$

14. $\int \frac{dx}{\sqrt{3x^2+4x+5}}$

B) Evaluate the following integrals:

1. $\int \frac{1}{5-4\cos x}\,dx$

2. $\int \frac{dx}{2-3\sin 2x}$

3. $\int \frac{dx}{4+5\sin x}$

4. $\int \frac{dx}{1-\cos x+\sin x}$

5. $\int \frac{dx}{4\sin x+3\cos x+13}$

6. $\int \frac{dx}{5+4\cos x}$

7. $\int \frac{dx}{3\sin x+4\cos x}$

8. $\int \frac{2\sin x+3\cos x}{3\sin x+4\cos x}\,dx$

ANSWERS

A)

1. $\log\left(x-1+\sqrt{x^2-2x+5}\right)+C$

2. $\frac{1}{2}\left(\log(x^2-2x+5)+\tan^{-1}\frac{x-1}{2}\right)+C$

3. $\sqrt{x^2-2x+5}+\log\left(x-1+\sqrt{x^2-2x+5}\right)+C$

4. $\frac{-1}{2}\log(3+2x-x^2)+\frac{1}{2}\log\left(\frac{x+1}{3-x}\right)+C$

5. $3\sin^{-1}\left(\frac{x+3}{\sqrt{24}}\right)+C$

6. $\sqrt{2ax-x^2}+C$

7. $\log\left(x+1+\sqrt{x^2+2x}\right)+C$

8. $\sqrt{8+2x-x^2}+C$

9. $\sqrt{x^2+4x+5}-2\log\left(x+2+\sqrt{x^2+4x+5}\right)+C$

10. $-\log\left(\frac{1+\sqrt{x^2+6x+10}}{x+3}\right)+C$

11. $\frac{1}{4}\log(4x^2+4x+5)+\frac{1}{2}\tan^{-1}\left(\frac{2x+1}{2}\right)+C$

12. $\sqrt{x^2+4x+13}-2\log\left(\sqrt{x^2+4x+13}+x+2\right)+C$

13. $2x-\frac{9}{2}\log(x^2+6x+10)+11\tan^{-1}(x+3)+C$

14. $\frac{1}{\sqrt{3}}\log\left[\left(x+\frac{2}{3}\right)+\frac{\sqrt{3x^2+4x+5}}{\sqrt{3}}\right]+C$

B)

1. $\frac{2}{3}\tan^{-1}(3\tan x/2)+C$

2. $\frac{1}{2\sqrt{5}}\log\left[\frac{2\tan x-(3+\sqrt{5})}{2\tan x-(3-\sqrt{5})}\right]+C$

3. $\frac{1}{3}\log\left(\frac{1+2\tan x/2}{4+2\tan x/2}\right)+C$

4. $-\log\left(1+\cot\frac{x}{2}\right)+C$

5. $\frac{1}{6}\tan^{-1}\left(5/6\tan x/2+1/3\right)+C$

6. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan x/2\right)+C$

7. $\frac{1}{5}\log\left[\tan\left(\frac{x+\tan^{-1}\frac{4}{3}}{2}\right)\right]+C$

Integration by parts

We sometime we encounter with problems which can be only be evaluated by using method of substitution and as well as integration by parts (or partial fractions).

Derivation of Integration by Parts

The process of integration of the product of two functions is known as integration by parts.

Let u and v be two functions. Then from differential calculus, we have

$$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides w.r.t.x, we get

$$u.v = \int u.\frac{dv}{dx}\ dx + \int v.\frac{du}{dx}\ dx$$

Transposing, we get

$$\int u.\frac{dv}{dx}\ dx = uv - \int v.\frac{du}{dx}\ dx \ldots\ldots\ldots\ldots\ldots(1)$$

Putting, $u = f(x)$

and $\frac{dv}{dx} = g(x)$

we get,

$$v = \int g(x)\,dx$$

So (1) be written as

$$\int f(x).g(x)\,dx = f(x)\int g(x)dx - \int \frac{df(x)}{dx}\left\{\int g(x)dx\right\}dx$$

This rule can be remembered as

$$\text{1st} \times \text{2nd} = \text{1st}\int \text{2nd} - \int[d(\text{1st})\int \text{2nd}].$$

Remark:

1. There are no general rules for choosing the first and second function. Yet the following rule works in most of the cases.

<u>Rule</u>: ILATE

We remember this, where

I = Inverse circular function

L = logarthmic function

A = Algebraic function

T = Trigonometric function

E = Exponential function.

In the product of two functions, we choose former in the above rule as the first function and the latter as the second function.

For example,

$$\int x^2 \sin x\,dx$$

In this integral, there are two functions algebraic and trigonometric. In the rule ILATE, algebraic is former and trigonometric is latter. So, we choose algebraic function as the first and trigonometric as the second and write $\int x^2 \sin x\,dx = x^2\int \sin x\,dx - \int\frac{d}{dx}(x^2).\{\int \sin x\,dx\}\,dx$ and evaluate the integral.

2. The integration by parts $\int uv\,dx$ can be written as

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$, where dash and suffix represents differentiation and integration with respect to x.

But this formula is not applicable for all cases.

Example 1.

Evaluate: $\int xe^x\,dx$.

Solution:

Let $I = \int x\,e^x\,dx$

We have two functions, x algebraic function and e^x(exponential function), (So by ILATE) choosing x as the first function and e^x as the second function and integrating by parts,

$$I = x\int e^x\,dx - \int\frac{d}{dx}(x)\ \left\{\int e^x\,dx\right\}dx$$

$$= x\,e^x - \int 1\,e^x dx = x\,e^x - e^x + C = e^x(x-1)+C$$

Example 2.

Evaluate: $\int x^3\cos(x^2)dx$

Solution:

Since angle of cosine is x^2, we first need to put $x^2 = t$.

$$2x\,dx = dt$$

$$\therefore \quad x\,dx = \frac{dt}{2}$$

$$\therefore \quad \text{Then, } I = \int x^2\cos(x^2)x\,dx = \int t\cos t\frac{dt}{2}$$

Now integrating by parts $= \frac{1}{2}[t.\sin t - \int \sin t.1dt] = \frac{1}{2}[t.\sin t + \cos t]$

Replacing t by x^2,

$$I = \frac{1}{2}[x^2\sin(x^2) + \cos(x^2)] + c$$

Example 3.

$$\int \tan^{-1}x\,dx$$

Solution:

$$\text{Let } I = \int \tan^{-1}x\,dx = \int 1\tan^{-1}x\,dx$$

Now, choosing $\tan^{-1}x$ as the first and 1 as the second function and integrating by parts, we get

$$I = \tan^{-1}x\int 1\,dx - \int \frac{d}{dx}(\tan^{-1}x)\ \{\int 1\,dx\}\,dx$$

$$= \tan^{-1}x\ x - \int \frac{1}{1+x^2}\ x\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$$

Example 4.

Integrate: $\int x\sin^2 nx\,dx$

Solution:

$$\text{Let } I = \int x\sin^2 nx\,dx$$

$$= \int x\left(\frac{1-\cos 2nx}{2}\right)dx = \frac{1}{2}[\int x\,dx - \int x\cos 2nx\,dx]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \left\{x\frac{\sin 2nx}{2n} + 1\frac{\cos 2nx}{(2n)^2}\right\}\right] + C$$

$$\therefore \quad I = \frac{1}{2}\left[\frac{x^2}{2} - \frac{x\sin 2nx}{2n} - \frac{1}{4n^2}\cos 2nx\right] + C$$

Example 5.

$$\int x\cos 2x\,dx$$

Solution:

$$\text{Let } I = \int x\cos 2x\,dx$$

Choosing x as the first and cos 2x as the second function and integrating by parts, we get

$$I = x\int \cos 2x\,dx - \int \frac{d}{dx}(x)\ \{\int \cos 2x\,dx\}\,dx$$

$$= x\frac{\sin 2x}{2} - \int 1\frac{\sin 2x}{2}\,dx$$

$$= \frac{x}{2}\sin 2x + \frac{1}{2}\frac{\cos 2x}{2} + C$$

$$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C.$$

Example 6.

Evaluate: $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

$$\text{Let } I = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

Solution:

$$\text{Let } \sin^{-1}x = \theta$$

Diff. w.r.t. x, we get

$$\frac{d}{dx}(\sin^{-1}x) = \frac{d\theta}{dx}$$

$$\text{or, } \frac{1}{\sqrt{1-x^2}}\,dx = d\theta$$

Also, from $\sin^{-1}x = \theta$, we get

$$x = \sin\theta$$

$$\text{So, } I = \int \theta \sin\theta \, d\theta$$

Now, integrating by parts, we get

$$\theta \int \sin\theta \, d\theta - \int \frac{d}{d(\theta)}(\theta) \int \sin\theta d\theta$$

$$= \theta(-\cos\theta) - \int 1(-\cos\theta)d\theta$$

$$= -\theta\cos\theta + \int \cos\theta \, d\theta$$

$$= -\theta\cos\theta + \sin\theta + C.$$

Substituting the value of θ, we get

$$I = -\sin^{-1}x \sqrt{1-x^2} + x + C$$

Example 7.

Evaluate: $\int e^{ax} \sin bx \, dx$

Solution:

Let $I = \int e^{ax} \sin bx \, dx$

Now, integrating by parts taking sinbx as first and e^{ax} as second function, we get

$$I = \sin bx \int e^{ax} dx - \int\left[\frac{d}{dx}(\sin bx)\int e^{ax}dx\right]dx$$

$$= \frac{e^{ax}}{a} \sin bx - b\int \cos bx \frac{e^{ax}}{a} dx$$

Again, integrating second expression by parts, we get

$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left[\cos bx \int e^{ax}dx - \int\left\{\frac{d}{dx}(\cos bx)\int e^{ax}dx\right\}dx\right]$$

$$\text{or, } I = \frac{e^{ax}}{a} \sin bx - \frac{b}{a}\left[\cos bx \frac{e^{ax}}{a} - \int b(-\sin bx)\frac{e^{ax}}{a}dx\right]$$

$$\text{or, } I = \frac{e^{ax}}{a} \sin bx - \frac{b}{a^2} \cos bx \, e^{ax} - \frac{b^2}{a^2}\int e^{ax} \sin bx \, dx$$

$$\text{or, } I = \frac{e^{ax}}{a} \sin bx - \frac{b}{a^2} \cos bx \, e^{ax} - \frac{b^2}{a^2} I$$

$$\text{or, } \left(1+\frac{b^2}{a^2}\right) I = e^{ax}\left[\frac{a \sin bx - b\cos bx}{a^2}\right] + C$$

$$\text{or, } I\left(\frac{a^2+b^2}{a^2}\right) = e^{ax}\left[\frac{a \sin bx - b\cos bx}{a^2}\right] + C \quad \left[\because c = \frac{k}{(a^2+b^2)}\right]$$

$$\therefore I = e^{ax}\left[\frac{a \sin bx - b\cos bx}{a^2+b^2}\right] + C$$

$$\text{Similarly, } \int e^{ax} \cos bx \, dx = e^{ax}\left[\frac{a \cos bx + b\sin bx}{a^2+b^2}\right] + C$$

Different cases of integration by parts:

Case I Integral of the function of the type

$$\int e^x[f(x)+f'(x)] \, dx = e^x f(x)+C$$

Proof:

$$\text{L.H.S.} = \int e^x[f(x)+f'(x)] \, dx$$

$$= \int e^x f(x) \, dx + \int e^x f'(x) \, dx$$

$$= \int e^x f(x) \, dx + \left\{e^x \int (f'(x) \, dx) - \int \frac{d}{dx}(e^x) \int f'(x) \, dx\right\}$$

To make comparable with ILATE, integrate the first integral taking e^x as second function.

$$= \int e^x f(x) \, dx + e^x f(x) - \int e^x f(x) \, dx + c$$

$$= e^x f(x) + C$$

= RHS Proved.

Example 8.

Evaluate $\int e^x(\sin x+\cos x) \, dx$

Solution:

We have, $\int e^x\{f(x)+f'(x)\} \, dx = e^x f(x) + C$

Then, $\int e^x(\sin x+\cos x) \, dx = e^x \sin x + C$

Example 9.

Evaluate $\int \frac{xe^x}{(x+1)^2}\,dx$

Solution:

We have, $\int \frac{xe^x}{(x+1)^2}\,dx = \int \frac{\{(x+1)-1\}e^x}{(x+1)^2}\,dx = \int \left\{\frac{1}{x+1} - \frac{1}{(x+1)^2}\right\} e^x\,dx$

$= \int \{f(x)+f'(x)\}\, e^x\,dx$ where $f(x) = \frac{1}{x+1}$

$= e^x f(x)+C$

$= e^x \frac{1}{x+1} + C.$

Case II

Integral of the functions of the form:

(i) $\int \sqrt{a^2-x^2}\,dx$

Solution:

Let $I = \int \sqrt{a^2-x^2}\,dx = \int 1\sqrt{a^2-x^2}\,dx$

Integrating by parts, taking $\sqrt{a^2-x^2}$ as the first function, we get

$I = \sqrt{a^2-x^2}\int 1\,dx - \int \frac{d}{dx}(a^2-x^2)^{1/2}\left(\int 1\,dx\right)dx$

$= \sqrt{a^2-x^2}\; x - \int \frac{1}{2}(a^2-x^2)^{-1/2}(-2x)\; x\,dx$

$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx$

$= x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \int \sqrt{a^2-x^2}\,dx$

$= x\sqrt{a^2-x^2} + a^2 \sin^{-1}(x/a) - I$

or, $2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}(x/a) + C$

$\therefore I = x/2\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + C.$

(ii) Evaluate: $\int \sqrt{x^2-a^2}\,dx$

Proceeding as above, we get

$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)+C$

(iii) Evaluate: $\int \sqrt{x^2+a^2}\,dx$

Similarly as above, we get

$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)+C$

Example

Evaluate: $\int \sqrt{\frac{a+x}{x}}\,dx$

Solution:

Let $I = \int \sqrt{\frac{a+x}{x}}\,dx$

Put $x = a\tan^2\theta$

$dx = 2a\tan\theta\sec^2\theta\,d\theta$, then

$I = \int \sqrt{\frac{a+a\tan^2\theta}{a\tan^2\theta}} \times 2a\tan\theta\sec^2\theta\,d\theta$

$= \int \sqrt{\frac{\sec^2\theta}{\tan^2\theta}} \times 2a\tan\theta\sec^2\theta\,d\theta$

$= 2a\int \sec^2\theta.\sec\theta\,d\theta$

$= 2a\int \sqrt{\tan^2\theta+1}\,\sec^2\theta\,d\theta$

Put $\tan\theta = t$

$\sec^2\theta\,d\theta = dt$

$= 2a\int \sqrt{t^2+1}\,dt = 2a\left[\frac{t\sqrt{t^2+1}}{2} + \frac{1}{2}\log\left(t+\sqrt{t^2+1}\right)\right]+C$

$= 2a\left[\frac{\tan\theta\sqrt{1+\tan^2\theta}}{2} + \frac{1}{2}\log\left(t+\sqrt{t^2+1}\right)\right]+C$

$\therefore I = 2a\left[\sqrt{\frac{x}{a}}.\sqrt{1+\frac{x^2}{a^2}} + \log\left(\sqrt{\frac{x}{a}}+\sqrt{1+\frac{x^2}{a^2}}\right)\right]+C$

Note: This problem can also be solved by putting $x = t^2$.

EXERCISE 11.4

Evaluate the integrals:

1. $\int x \sin x \, dx$
2. $\int x^2 \sin x \, dx$
3. $\int x \log x \, dx$
4. $\int x^n \log ax \, dx, n \neq -1$
5. $\int \log x \, dx$
6. $\int x^5 e^x \, dx$
7. $\int x^5 \sin x \, dx$
8. $\int \frac{xe^x}{(x+1)^2} dx$
9. $\int \frac{\log x}{x^2} dx$
10. $\int \sin^{-1} x \, dx$
11. $\int x \sin x \sin 2x \sin 3x \, dx$
12. $\int \frac{x+\sin x}{1+\cos x} dx$
13. $\int e^x(\tan x - \log \cos x) \, dx$
14. $\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}} dx$
15. $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$
16. $\int x\cos^3 x \sin x \, dx$
17. $\int \cos(\log x) \, dx$
18. $\int e^x \sin x \, dx$
19. $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}} dx$
20. $\int \sin\sqrt{x} \, dx$

ANSWERS

1. $-x \cos x + \sin x + C$
2. $-x^2\cos x + 2x\sin x + 2\cos x + C$
3. $\frac{x^2(\log x)}{2} - \frac{x^2}{4} + C$
4. $\frac{x^{n+1}}{n+1}\left[\log ax - \frac{1}{(n+1)}\right] + C$
5. $(x \log x - x) + C$
6. $(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) e^x + C$
7. $(-x^5 + 20x^3 - 120x) \cos x + (5x^4 - 60x^2 + 120) \sin x + C$
8. $\frac{e^x}{x+1} + C$
9. $-\frac{1}{x}(\log x + 1) + C$

10. $x \sin^{-1} x + \sqrt{1-x^2} + C$
11. $-\frac{x}{8}(\cos 2x + \frac{1}{2}\cos 4x - \frac{1}{3}\cos 6x) + \frac{1}{16}\left(\sin 2x + \frac{1}{4}\sin 4x - \frac{1}{9}\sin 6x\right) + C$
12. $x \tan\frac{x}{2} + C$
13. $e^x . \log(\sec x) + C$
14. $\frac{1}{\sqrt{1+x^2}}(x - \tan^{-1}x) + C$
15. $2x \tan^{-1}x - \log(1+x^2) + C$
16. $\frac{-x}{4}\cos^4 x + \frac{1}{128}(12x + 8\sin 2x + \sin 4x) + C$
17. $\left(\frac{x}{2}\right)[\cos(\log x) + \sin(\log x)] + C$
18. $\frac{e^x}{2}(\sin x - \cos x) + C$
19. $\frac{1}{2}\left(x \cos^{-1}x - \sqrt{1-x^2}\right) + C$
20. $2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}) + C$

Integration by partial fraction

Since every polynomial with real coefficients can be expressed as a product of real factors and real irreducible quadratic factors, hence the rational algebraic fraction can thus be resolved into partial fractions.

There are four cases depending on the nature of the factors in the denominator.

Use of partial fraction: There are some integrand which can only be evaluated by using partial fraction. For example $\int \frac{x+1}{(x-1)(x-2)(x-3)} dx$.

To evaluate it let

$$\frac{x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

or, $x + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

Case I: Distinct linear factors:

Example 1.

Evaluate $\int \frac{x}{(x-3)(x+1)} dx$

Solution:

Let $\frac{x}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$

$$\text{or, } x = A(x+1) + B(x-3)$$

Now, Putting $x = -1$, we get

$$-1 = B - 4$$

$$\therefore B = \frac{1}{4}.$$

Again, Putting $x = 3$, we get

$$3 = 4A$$

$$\therefore A = \frac{3}{4}.$$

Now,

$$I = \int \frac{x}{(x-3)(x+1)}\,dx = \int\left(\frac{3}{4}\cdot\frac{1}{(x-3)} + \frac{1}{4}\cdot\frac{1}{x+2}\right)dx$$

$$= \frac{3}{4}\int \frac{1}{x-3}\,dx + \frac{1}{4}\int\frac{1}{x+1}\,dx$$

$$= \frac{3}{4}\log(x-3) + \frac{1}{4}\log(x+1) + C$$

Example 2.

Evaluate: $\int \frac{x+2}{x^2 - 13x + 42}dx$

Solution:

Let $I = \int \frac{x+2}{x^2-13x+42}dx$ .... (i)

Here, $\frac{x+2}{x^2-13x+42} = \frac{x+2}{(x-6)(x-7)} = \frac{A}{x-6} + \frac{B}{x-7}$

This gives, $x+2 = A(x-7) + B(x-6)$

Put $x = 6$, we get, $8 = -1\,A \Rightarrow A = -8$

Put $x = 7$, we get, $9 = B$

Thus we get,

$$I = \int\left[\frac{-8}{x-6} + \frac{9}{x-7}\right]dx$$

$$= 9\int\frac{dx}{x-7} - 8\int\frac{dx}{x-6}$$

$$= 9\log(x-7) - 8\log(x-6) + c$$

$$= \log(x-7)^9 - \log(x-6)^8 + c$$

$$\therefore I = \log\left[\frac{(x-7)^9}{(x-6)^8}\right] + C$$

Case II: Repeated linear factors:

Example 3.

Evaluate $\int \frac{x+5}{(x+1)(x+2)^2}dx.$

Solution:

Let $\frac{x+5}{(x+1)(x+2)^2} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$

$$\text{or, } x + 5 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

Putting $x = 2$ and $x = -1$, we get

$$A = 4,\ C = -3$$

Comparing co–efficient of x^2, we get

$$A + B = 0$$

$$\text{i.e. } B = -A = -4.$$

$$\therefore I = \int\frac{x+5}{(x+1)(x+2)^2}dx = 4\int\frac{dx}{x+1} - 4\int\frac{dx}{x+2} - 3\int\frac{dx}{(x+2)^2}$$

$$= 4\log(x+1) - 4\log(x+2) + \frac{3}{(x+2)} + C$$

$$= 4\log\left(\frac{x+1}{x+2}\right) + \frac{3}{x+2} + C.$$

Case III: Repeated quadratic factors:

Example 4.

Integrate $\frac{2}{(x^2+1)(x^2+3)}$ w.r.t.x.

Solution:

Let $\frac{2}{(x^2+1)(x^2+3)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{x^2+3}$

$$\text{or, } 2 = (Ax+B)(x^2+3) + (Cx+D)(x^2+1)$$

Comparing Coefficients of x^3, x^2, x and the constant terms, we get

$A+C = 0$ – (i) (Coefficients of x^3)

$B+D = 0$ – (ii) (Coefficients of x^2)

$3A+C = 0$ – (iii) (Coefficients of x)

And, $3B+D = 2$ – (iv) (Coefficients of constant)

From (i) and (iii) we get

$C = 0$ and $A = 0$

Also, From (ii) and (iv) we get

$B = 1, D = -1$

$$\frac{2}{(x^2+1)(x^2+3)} = \frac{1}{x^2+1} - \frac{1}{x^2+3}$$

So,

$$I = \int \frac{2}{(x^2+1)(x^2+3)} dx = \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+3}\right) dx$$

$$= \int \frac{dx}{x^2+1} - \int \frac{dx}{x^2+3}$$

$$\therefore \quad I = \tan^{-1} x - \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C$$

Case IV: Improper fraction

When the power of the variable in numerator is equal to or greater than the power of variable in denominator:–

In this case, we first divide the numerator by denominator and make the power of numerator smaller than that of the denominator. Then follow the usual process.

Example 5.

$$\int \frac{x^3}{(x-2)(x-3)} dx.$$

Solution:

Now, by actual division, we get

$$\frac{x^3}{(x-2)(x-3)} = x+5+\frac{19x-30}{(x-2)(x-3)}$$

So,

$$I = \int \frac{x^3}{(x-2)(x-3)} = \int \left\{x+5+\frac{19x-30}{(x-2)(x-3)}\right\} dx$$

$$= \int x\,dx + 5\int dx + \int \frac{19x-30}{(x-2)(x-3)} dx$$

$$= \int x\,dx + 5\int dx + \int \frac{19x-30}{(x-2)(x-3)} dx.$$

Now, Let, $I_1 = \int \frac{19x-30}{(x-2)(x-3)} dx$

Now,

Let, $\frac{19x-30}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$

or, $19x - 30 = A(x-3)+B(x-2)$

Putting $x = 3$, we get

$B = 27$

Putting $x = 2$, we get

$A = -8$

$$\therefore I_1 = \int \frac{-8}{(x-2)} dx + \int \frac{27}{(x-3)} dx = -8 \log (x-2) + 27 \log (x-3)$$

$$\therefore I = x^2/2 + 5x - 8 \log (x-2) + 27 \log (x-3) + C$$

Example 6.

$$\int \frac{\cos x}{(1+\sin x)(2+\sin x)} dx,$$

Solution:

We first put $\sin x = t$

$\cos x\, dx = dt$

$\therefore$

Then $I = \int \frac{dt}{(1+t)(2+t)} = \int \left[\frac{1}{1+t} - \frac{1}{2+t}\right] dt$ by using partial fraction

$= \log(1 + t) - \log(2 + t) + c$

$$= \log \frac{1+t}{2+t} + c = \log \frac{1+\sin x}{2+\sin x} + c$$

EXERCISE 11.5

Integrate the following functions w.r.t. x:

1. (i) $\frac{x}{(x-3)(x+1)}$ (ii) $\frac{5x-3}{(x+1)(x-3)}$

(iii) $\frac{(x-1)(x-2)}{(x+3)(x+4)(x+5)}$ (iv) $\frac{x+12}{(x^2-13x+42)}$

2. (i) $\frac{x+4}{(x+1)^2}$ (ii) $\frac{x^2-4}{(x^2+1)(x^2+3)}$

(iii) $\frac{x^2}{(x-a)(x-b)}$ (iv) $\frac{2x}{(x^2+1)(x^2+3)}$

3. (i) $\dfrac{1}{1+3e^x+2e^{2x}}$ (ii) $\dfrac{x}{(x-1)^2(x+2)}$

(iii) $\dfrac{x^2+8}{x^2-5x+6}$ (iv) $\dfrac{1}{x(x^2+x+1)}$

4. (i) $\dfrac{x^2}{(x^2+1)(x^2+4)}$ (iv) $\dfrac{x^3-5x}{(x^2-9)(x^2+1)}$

(ii) $\dfrac{1}{x[6(\log x)^2+7\log x+2]}$ (v) $\dfrac{1}{(e^x-1)^2}$

(iii) $\dfrac{\cos x}{(\sin x+2)(2\sin x+3)}$ (vi) $\dfrac{7x^2+3x+1}{x^2+x}$

ANSWERS

1. (i) $\frac{1}{4}\log[(x-3)^3(x+1)]+C$

(ii) $3\log(x-3)+2\log(x+1)+C$

(iii) $10\log(x+3)-30\log(x+4)+21\log(x+5)+C$

(iv) $19\log(x-7)-18\log(x-6)+C$

2. (i) $\log(x+1)-\dfrac{3}{(x+1)}+C$

(ii) $\dfrac{7}{2\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right)\dfrac{-5}{2}\tan^{-1}(x)+C$

(iii) $x+\dfrac{a^2}{a-b}\log(x-a)-\dfrac{b^2}{a-b}\log(x-b)+C$

(iv) $\dfrac{1}{2}\log\left(\dfrac{x^2+1}{x^2+3}\right)+C$

3. (i) $x+\log(e^x+1)-2\log(1+2e^x)+C$

(ii) $\dfrac{2}{9}\log\dfrac{x-1}{x+2}-\dfrac{1}{3(x-1)}+C$

(iii) $x-12\log(x-2)+17\log(x-3)+C$

(iv) $\dfrac{1}{2}\log\left(\dfrac{x}{x^2+x+1}\right)-\left(\dfrac{1}{\sqrt{3}}\right)\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+C$

4. (i) $\dfrac{-1}{3}\tan^{-1}(x)+\dfrac{2}{3}\tan^{-1}\left(\dfrac{x}{2}\right)+C$

(ii) $\log\dfrac{2\log x+1}{3\log x+2}+C$ (iii) $\log\left(\dfrac{2\sin x+3}{\sin x+2}\right)+C$

(iv) $\dfrac{1}{5}\log(x^2-9)+\dfrac{3}{10}\log(x^2+1)+C$

(v) $\log\left(\dfrac{e^x}{e^x-1}\right)-\dfrac{1}{e^x-1}+C$

(vi) $7x+\log x-5\log(x+1)+C$

Miscellaneous Exercise

1. Show that $\int\dfrac{2\sin x+3\cos x}{3\sin x+4\cos x}dx=\dfrac{1}{25}\log(3\sin x+4\cos x)+\dfrac{18}{25}x+c$

2. Show that $\int\dfrac{e^x(1+\sin x)dx}{1+\cos x}=e^x\tan\dfrac{x}{2}+c$

3. Show that : $\int\sqrt{\dfrac{\sin(x-\alpha)}{\sin(x+\alpha)}}dx$

$=\cos\alpha\cos^{-1}(\cos x\sec\alpha)-\sin\alpha\log(\sin x+\sqrt{\sin^2x-\sin^2\alpha})+c$

4. Show that $\int\dfrac{dx}{x\sqrt{x^4-1}}=\dfrac{1}{2}\sec^{-1}(x^2)+c$

5. Show that $\int\dfrac{x^2+1}{x^4+1}dx=\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{x^2-1}{x\sqrt{2}}\right)+c$

6. Show that $\int\dfrac{\sin 2x\,dx}{a\sin^2x+b\cos^2x}=\dfrac{1}{a-b}\log(a\sin^2x+b\cos^2x)+c$

7. Show that $\int\dfrac{dx}{\sqrt{1+\sin x}}=\sqrt{2}\log\tan\left(\dfrac{x}{4}+\dfrac{\pi}{8}\right)+c$

8. Show that $\int\dfrac{\sin x\,dx}{\sqrt{1+\sin x}}=2\sqrt{1-\sin x}-\sqrt{2}\log\tan\left(\dfrac{x}{4}+\dfrac{\pi}{8}\right)+c$

9. Show that $\int\dfrac{(1+\cos x)}{\sin x.\cos x}dx=\log\left(\tan x.\tan\dfrac{x}{2}\right)+c$

10. Show that $\int\dfrac{dx}{a^2-b^2\cos^2x}=\dfrac{1}{a\sqrt{a^2-b^2}}\tan^{-1}\left(\dfrac{a}{\sqrt{a^2-b^2}}\tan x\right)$ for $a>b$.

11. Show that $\int\dfrac{dx}{a^2\cos^2x+b^2\sin^2x}=\dfrac{1}{ab}\tan^{-1}\left(\dfrac{b}{a}\tan x\right)+c$

12. Show that $\int\dfrac{\log x}{(1+\log x)^2}dx=\dfrac{x}{1+\log x}+c$

13. Show that $\int \sqrt{\sec x - 1}\, dx = -2\log\left[\cos\frac{x}{2} + \sqrt{\frac{\cos x}{2}}\right] + c$

14. Show that $\int \frac{5\cos x + 6}{2\cos x + \sin x + 3}\, dx = 2x + \log(2\cos x + \sin x + 3) + c$

15. Show that $\int \frac{dx}{\sin x(3 + 2\cos x)}$

$$= -\frac{1}{2}\log(1+\cos x) + \frac{1}{10}\log(1-\cos x) + \frac{2}{5}\log(3+2\cos x) + c$$

16. Show that $\int \frac{(1+\sin x)dx}{\sin x(1+\cos x)} = \frac{1}{2}\log\left(\tan\frac{x}{2}\right) + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2} + c$

Definite Integral

In geometrical and other applications of Integral calculus, it is necessary to find the difference in the value of the integral of a function f(x) between two values of an independent variable x, say a and b. The difference in the values is called the definite integral of f(x), over the interval [a, b] and is denoted by $\int_a^b f(x)\, dx$.

Thus $\int_a^b f(x)\, dx = F(b) - F(a)$, where F(x) is an integral of f(x).

The difference F(b)–F(a) is also denoted by $[F(x)]_a^b$.

Here, the values a and b are called the lower limit and upper limit respectively. As,

$$\int_a^b f(x)\, dx = [F(x)+c]_b^a$$

$$= [\{F(b)+c\} - \{F(a)+c\}] = F(b) - F(a).$$

Thus, the arbitrary constant doesn't appear and the integral is called definite.

The rule of evaluating a definite integral:

(i) Find the indefinite integral.

(ii) Calculate its value when x = b.

(iii) Calculate its value when x = a.

(iv) Subtract the value in (iii) from the value in (ii).

Example 1.

$$I = \int_2^4 x^3\, dx = \left[\frac{x^4}{4}\right]_2^4 = \frac{(4)^4}{4} - \frac{2^4}{4} = (64-4)$$

$$= 60.$$

Example 2.

$$I = \int_0^1 \frac{1-x^2}{1+x^2}\, dx = \int_0^1 \frac{2-(1+x^2)}{1+x^2}\, dx = \int_0^1 \left(\frac{2}{1+x^2} - 1\right) dx$$

$$= \left[2\tan^{-1}x - x\right]_0^1 = 2\tan^{-1}(1) - 1 - 2\tan^{-1}(0) + 0$$

$$= 2.\pi/4 - 1 = \pi/2 - 1$$

Note: To solve this problem we may put x = tanθ.

Example 3.

$$I = \int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$$

Solution:

Put $x = \sec\theta$

$dx = \sec\theta \tan\theta\, d\theta$

When $x = 1, \theta = 0$ and $x = 2, \theta = \frac{\pi}{3}$

Then,

$$I = \int_0^{\frac{\pi}{3}} \frac{1}{1+\cos\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{2}\sec^2\frac{\theta}{2}\, d\theta = \left[\tan\frac{\theta}{2}\right]_0^{\frac{\pi}{3}} = \frac{1}{\sqrt{3}} - 0 = \frac{1}{\sqrt{3}}$$

OR

Put $x+1 = \frac{1}{t}$, Then $dx = \frac{-1}{t^2}\, dt$

When $x = 1, t = \frac{1}{1+x} = \frac{1}{1+1} = \frac{1}{2}$

When $x = 2,\ t = \frac{1}{1+x} = \frac{1}{1+2} = \frac{1}{3}$

$$\therefore I = \int_{1/2}^{1/3} \frac{-1/t^2 . dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2 - 1}} = -\int_{1/2}^{1/3} \frac{dt}{\sqrt{(1-t)^2 - t^2}}$$

$$= -\int_{1/2}^{1/3} \frac{dt}{\sqrt{1-2t}} = -\left[\frac{(1-2t)^{1/2}}{\frac{1}{2}(-2)}\right]_{1/2}^{1/3} = \left[(1-2t)^{1/2}\right]_{1/2}^{1/3}$$

$$= \left(1-2\times\frac{1}{3}\right)^{1/2} - \left(1-2\times\frac{1}{2}\right)^{1/2} = \frac{1}{\sqrt{3}}.$$

Properties of definite integral

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$

i.e. Definite integral is independent of the variable chosen.

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (a < c < b)$

4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

5. $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$, when f(x) is even function

$= 0$, when f(x) is odd function.

Note: A function f(x) is said to be an odd function if $f(-x) = -f(x)$, for all values of x. Also a function g (x) is said to be an even function if $g(-x) = g(x)$ for all x.

$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, when $f(2a-x) = f(x)$

$= 0$, when $f(2a-x) = -f(x)$

$\int_0^{na} f(x)dx = n\int_0^a f(x)dx$, when $f(a+x) = f(x)$

Example 1.

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

Solution:

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \text{..............(i)}$$

$$\text{or, } I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)}+\sqrt{\cos(\pi/2 - x)}}\, dx \qquad \text{(Using prop. 4)}$$

$$\text{or, } I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \text{..............(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right) dx = \int_0^{\pi/2} dx.$$

$$\text{or, } 2I = [x]_0^{\pi/2}$$

or, $2I = \pi/2$

$\therefore I = \pi/4$.

Example 2.

Show that $\int_0^{\pi/2} \log \sin\theta \, d\theta = \int_0^{\pi/2} \log \cos\theta d\theta = \frac{\pi}{2} \log \frac{1}{2}$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \log \sin\theta . d\theta \quad \ldots\ldots\ldots\ldots\ldots(i)$$

$$\text{or, } I = \int_0^{\pi/2} \log \left[\sin\left(\left(\pi/2 - \theta\right)\right)\right] d\theta$$

$$\therefore I = \int_0^{\pi/2} \log \cos\theta . d\theta \quad \ldots\ldots\ldots\ldots\ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log \sin\theta d\theta + \int_0^{\pi/2} \log \cos\theta . d\theta$$

$$= \int_0^{\pi/2} (\log \sin\theta + \log \cos\theta) = \int_0^{\pi/2} \log(\sin\theta . \cos\theta) \, d\theta$$

$$= \int_0^{\pi/2} \log \frac{2 \sin\theta . \cos\theta}{2} d\theta = \int_0^{\pi/2} [\log(\sin 2\theta) - \log 2] \, d\theta$$

$$= \int_0^{\pi/2} \log(\sin 2\theta) . d\theta - \log 2 \int_0^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \log(\sin 2\theta) d\theta - \log 2 [\theta]_0^{\pi/2}$$

$$2I = \int_0^{\pi/2} \log(\sin 2\theta) d\theta - \pi/2 \log 2. \ \ldots\ldots (i)$$

Putting $2\theta = t$, we get

$$d\theta = \frac{dt}{2}.$$

When $\theta = 0, t = 0$

When $\theta = \pi/2, t = \pi$

$$\int_0^{\pi/2} \log \sin 2\theta . d\theta = \frac{1}{2} \int_0^{\pi} \log \sin t . dt$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin t . dt = I, \text{ so from (i)}$$

$$\therefore 2I = I - \frac{\pi}{2} \log 2.$$

$$\therefore I = -\frac{\pi}{2} . \log 2 = \frac{\pi}{2} \log \frac{1}{2}$$

Thus we get,

$$\int_0^{\pi/2} \log \sin\theta \, d\theta = \int_0^{\pi/2} \log \cos\theta . d\theta = \frac{\pi}{2} \log \frac{1}{2}$$

Example 3.

$$\int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1 + x^2}$$

Solution:

$$I = \int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1 + x^2}$$

Put $x = \tan\theta$, Then $\frac{dx}{d\theta} = \sec^2\theta$

i.e., $dx = \sec^2\theta d\theta$

Also,

When $x \to 0$, $\theta \to 0$

When $x \to \infty$, $\theta \to \pi/2$

So,

$$I = \int_0^{\pi/2} \log\left(\tan\theta + \frac{1}{\tan\theta}\right)\frac{\sec^2\theta.d\theta}{1+\tan^2\theta}$$

$$= \int_0^{\pi/2} \log\left(\frac{\sec^2\theta}{\tan\theta}\right) d\theta = \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} [\log 1 - \log(\sin\theta\cos\theta)]\, d\theta$$

$$= \int_0^{\pi/2} (0 - \{\log(\sin\theta) + \log(\cos\theta)\})\, d\theta$$

$$= -\left[\int_0^{\pi/2} \log\sin\theta.d\theta + \int_0^{\pi/2} \log\cos\theta.d\theta\right]$$

$$= -\left[\frac{\pi}{2}\log {}^1/_2 + {}^{\pi}/_2 \log\frac{1}{2}\right]$$

$$= -\pi\log\frac{1}{2} \qquad \left(\because \int_0^{\pi/2} \log\sin\theta = \int_0^{\pi/2} \log\cos\theta = {}^{\pi}/_2 \log\frac{1}{2}\right)$$

$\therefore$ $I = \pi\log 2$

Example 4.

Evaluate: $\int_1^4 \frac{x+2}{\sqrt{4x-x^2}}dx$

Solution:

$$\text{Let } I = \int_1^4 \frac{x+2}{\sqrt{4x-x^2}}dx = \int_1^4 \frac{x+2}{\sqrt{4-(x-2)^2}}dx$$

$$= \int_1^4 \frac{x-2+4}{\sqrt{4-(x-2)^2}}dx$$

$$\therefore \quad I = \int_1^4 \frac{(x-2)dx}{\sqrt{4-(x-2)^2}} + 4\int_1^4 \frac{dx}{\sqrt{4-(x-2)^2}}$$

Put $x-2 = t \quad dx = dt$

When $x = 1, t = -1$
$x = 4, t = 2$

Then,

$$I = \int_{-1}^{2} \frac{tdt}{\sqrt{4-t^2}} + 4\int_{-1}^{2}\frac{dt}{\sqrt{4-t^2}}$$

$$= \left[-\frac{1}{2}.2\sqrt{4-t^2} + 4\sin^{-1}\frac{t}{2}\right]_{-1}^{2}$$

$$= \left[-\sqrt{4-t^2} + 4\sin^{-1}\frac{t}{2}\right]_{-1}^{2}$$

$$= \left[4\sin^{-1}(1) + \sqrt{3} - 4\left(-\frac{\pi}{6}\right)\right]$$

$$\therefore \quad I = \left[2\pi + \sqrt{3} + \frac{2}{3}\pi\right]$$

Example 5.

Evaluate: $\int_1^2 \sqrt{2x-x^2}\, dx$

Solution:

$$\text{Let } I = \int_1^2 \sqrt{2x-x^2}\, dx = \int_1^2 \sqrt{1-(x-1)^2}\, dx$$

Put, $x - 1 = t$, $dx = dt$

When, $x = 1, t = 0$ and $x = 2, t = 1$

Then,

$$I = \int_0^1 \sqrt{1-t^2}\, dt = \left[\frac{t\sqrt{1-t^2}}{2} + \frac{1}{2}\sin^{-1} t\right]_0^1$$

$$= \frac{1}{2}\sin^{-1}(1)$$

$$\therefore \quad I = \frac{\pi}{4}$$

Example 6.

Evaluate: $\int_0^a \frac{x^4 dx}{\sqrt{a^2-x^2}}$

Solution:

We have, $I = \int_0^a \frac{x^4 dx}{\sqrt{a^2 - x^2}}$

Put, $x = a\sin\theta$ $\quad dx = a\cos\theta\, d\theta$

When, $x = 0, \theta = 0$ and $x = a$, implies $\theta = \frac{\pi}{2}$

Then, $I = \int_0^{\frac{\pi}{2}} \frac{a^4 \sin^4\theta}{a\cos\theta d\theta} a\cos\theta\, d\theta = a^4 \int_0^{\pi/2} \sin^4\theta\, d\theta$

$$= a^4 \int_0^{\pi/2} \sin^2\theta(1-\cos^2\theta)d\theta$$

$$= a^4 \int_0^{\pi/2} \left[\sin^2\theta - \left(\frac{2\sin\theta\cos\theta}{2}\right)^2\right] d\theta$$

$$= a^4 \left[\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) - \frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4}\right)\right]_0^{\pi/2}$$

$$= a^4\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{8}\cdot\frac{\pi}{2}\right] = a^4\left[\frac{\pi}{4} - \frac{\pi}{16}\right]$$

$$\therefore \; I = \frac{3\pi}{16} a^4$$

EXERCISE 11.6

Show That

1. $\int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\sin\theta + \cos\theta} = \frac{\pi}{4}$

2. $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{4}$

3. $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}} = \frac{\pi}{4}$

4. $\int_0^{\pi/2} \frac{\sqrt{\cot x}\, dx}{1+\sqrt{\cot x}} = \frac{\pi}{4}$

5. $\int_0^{\pi} \frac{x\tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$

6. $\int_0^a \frac{\sqrt{x}\, dx}{\sqrt{x} + \sqrt{a-x}} = \frac{a}{2}$

7. $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2$

8. $\int_0^{\pi} \frac{x\sin x\, dx}{1+\cos^2 x} = \frac{\pi^2}{4}$

9. $\int_0^{\pi/2} \frac{x\, dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)$

10. $\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8}\log 2.$

11. Show that $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x.\cos x} dx = 0$

12. $\int_0^{\frac{\pi}{2}} \log(\tan x)\, dx = 0$

13. $\int_0^{\frac{\pi}{2}} \sin 2x(\log \tan x)\, dx = 0$

14. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$

15. $\int_0^{\pi} \frac{x\tan x\, dx}{\sec x + \cos x} = \frac{\pi^2}{4}$

16. $\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x\, dx}{\cos^4 x + \sin^4 x} = \frac{\pi^2}{16}$

17. $\int_0^1 \cot^{-1}(1 - x + x^2)\, dx = \frac{\pi}{2} - \log 2.$

MISCELLANEOUS EXERCISES

1. $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx$

2. $\int \frac{dx}{(e^x + e^{-x})^2}$

3. $\int \sqrt{\frac{a+x}{x}} dx$

9. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\cos x}} dx$

10. $\int_0^{\pi/2} \frac{dx}{2+\cos x}$

11. $\int_1^e \frac{dx}{x\sqrt{1-(\log x)^2}}$

4. $\int \frac{x^2-1}{x^4-x^2+1}dx$
5. $\int \frac{x^2}{x^4+1}dx$
6. $\int \frac{\cos x}{\cos x+\sin x}dx$
7. $\int \log(\sqrt{x^2+1})dx$
8. $\int \tan^{-1}(\sqrt{x+1})dx$
12. $\int \frac{x+1}{x^2(x-1)}dx$
13. $\int \frac{8}{x^4+2x^3}dx$
14. $\int x.\tan^2 x\, dx$
15. $\int (x+1)^2 e^x\, dx$
16. $\int \cos\sqrt{x}\, dx$

ANSWERS

1. $2(\sin x + x\cos\alpha) + C$
2. $\frac{-1}{2(1+e^{2x})} + C$
3. $\sqrt{x(x+a)} + a\log(\sqrt{x+a}+\sqrt{x}) + C$
4. $\frac{1}{2\sqrt{3}}\log\left(\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right) + C$
5. $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\log\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right) + C$
6. $\frac{1}{2}[x+\log(\sin x+\cos x)]+C$
7. $x\log\sqrt{x^2+1} - x + \tan^{-1}x + C$
8. $(x+2)\tan^{-1}(\sqrt{x+1} - \sqrt{x+1}) + C$
9. $2-\sqrt{2}\log(1+\sqrt{2} + C$
10. $\frac{\pi}{3\sqrt{3}}$
11. $\frac{\pi}{2}$
12. $\frac{1}{x} + 2\log\left(1-\frac{1}{x}\right) + C$
13. $\frac{-2}{x^2} + \frac{2}{x} + \log\left(\frac{x}{x+2}\right) + C$
14. $x\tan x - \frac{x^2}{2} + \log(\cos x) + C$
15. $(x^2+1)e^x + C$
16. $2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

Infinite (or improper) Integrals

Introduction:

In definite integral, we supposed that the range of integration is finite and the integrand is continuous in the range. If in an integral either the range is infinite or the integrand has an infinite discontinuity in the range, then the integral is said to be infinite integral or improper integral. These integrals occur in finding area, volumes and surface revolution between a plane curve and asymptote.

(A) Improper integral having limit of integration is infinite: The following are different types of improper integral with infinite range:

i) The integral of type $\int_a^\infty f(x)\,dx$, which can be defined by $\lim_{b\to\infty}\int_a^b f(x)\,dx$, provided f(x) is integrable in (a, b) and this limit exists.

ii) The integral of type $\int_{-\infty}^b f(x)\,dx$, which can be defined by $\lim_{a\to-\infty}\int_a^b f(x)\,dx$, provided f(x) is integrable in (a, b) and this limit exists.

iii) The integral of type $\int_{-\infty}^\infty f(x)\,dx$, can be written as $\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$; which can be expressed in the type (i) and (ii).

(B) Improper integral, where integrand infinitely discontinuous at a point.

i) If f(x) is infinitely discontinuous only at the end point a. That is, $\lim_{x\to a} f(x) \to \infty$, then $\int_a^b f(x)\,dx$ is defined as $\lim_{h\to 0}\int_{a+h}^b f(x)\,dx$, provided $h > 0$ and f(x) be integrable in (a + h, b) and this limit exists.

ii) If f(x) is infinitely discontinuous only at the end point b. That is $\lim_{x\to b} f(x) \to \infty$, then $\int_a^b f(x)\,dx$ is defined as

$$\lim_{h\to 0}\int_a^{b-h} f(x),\ h>0,$$ provided f(x) be integrable in (a, b – h) and this limit exists.

iii) If f(x) is infinitely discontinuous only at an internal point c. Then $\int_a^b f(x)\,dx = \lim_{h\to 0}\int_a^{c-h} f(x)\,dx + \lim_{h'\to 0}\int_{c+h'}^{b} f(x)\,dx$ when $h\to 0$ and $h'\to 0$ independently.

iv) If a and b are both points of infinite discontinuity, then $\int_a^b f(x)\,dx$ is defined as $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, when these two integrals exists, as defined above; where $a < c < b$.

Examples:

1. Evaluate the improper integral $\int_0^\infty e^{-x}\,dx$.

Solution:

Let the integral

$$I = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[\frac{e^{-x}}{-1}\right]_0^b = \lim_{b\to\infty}(1-e^{-b}) = 1$$

Since $\lim_{b\to\infty} e^{-b} = 0$

Thus we get, $\int_0^\infty e^{-x}\,dx = 1$

2. Evaluate the integral $\int_0^\infty \sin tx\,dx$, if it exists.

Solution:

The given integral can be written as

$$I = \int_0^\infty \sin tx\,dx$$

$$= \lim_{b\to\infty}\int_0^b \sin tx\,dx$$

$$= \lim_{b\to\infty}\left(-\frac{\cos tx}{t}\right)_0^b$$

$$= -\frac{1}{t}\lim_{b\to\infty}(\cos tb - \cos 0)$$

$$= \lim_{b\to\infty}\left(\frac{1-\cos bt}{t}\right)$$

Which does not exists. Thus the given improper integral does not exists.

3. Evaluate the improper integral $\int_{-\infty}^{\infty}\frac{dx}{x^3}$ if it exists.

Solution:

We have, the integrand $f(x) = \frac{1}{x^3}$, which does not exists at $x = 0$. So f(x) is infinitely discontinuous at $x = 0$.

Then, $I = \int_{-\infty}^{\infty}\frac{1}{x^3}\,dx$

$$= \int_{-\infty}^{0}\frac{1}{x^3}\,dx + \int_{0}^{\infty}\frac{1}{x^3}\,dx$$

$$= \lim_{a\to\infty}\left(\int_{-a}^{0-h}\frac{1}{x^3}\,dx + \int_{0+h}^{a}\frac{1}{x^3}\,dx\right)$$

$$= \lim_{a\to\infty}\lim_{h\to 0}\left[\left(-\frac{1}{2x^2}\right)_{-a}^{0-h} + \left(-\frac{1}{2x^2}\right)_h^a\right]$$

$$= -\frac{1}{2}\lim_{a\to\infty}\lim_{h\to 0}\left[\left(\frac{1}{h^2}-\frac{1}{a^2}\right)+\left(\frac{1}{a^2}-\frac{1}{h^2}\right)\right]$$

$$= -\frac{1}{2}\lim_{a\to\infty}\lim_{h\to 0}(0)$$

$$= 0$$

Thus, we get $\int_{-\infty}^{\infty} \frac{1}{x^3}\,dx = 0.$

4. Evaluate the improper integral $\int_0^1 \log x \; dx.$

Solution:

In the integral $\int_0^1 \log x\,dx$, the integrand $f(x) = \log x$, which does not exists at $x = 0$. So $x = 0$ is infinite discontinuous.

$$\text{Here, } I = \int_0^1 \log x\,dx = \lim_{h\to 0} \int_{0+h}^{1} \log x\,dx$$

$$= \lim_{h\to 0} (x\log x - x)_h^1 = \lim_{h\to 0} (h - h\log h - 1)$$

$$= -\left[\lim_{h\to 0} (h\log h) + 1\right] = \left(\lim_{h\to 0} \frac{\log h}{\frac{1}{h}} + 1\right)$$

$$= -\left[\lim_{h\to 0} \left(\frac{\frac{1}{h}}{-\frac{1}{h^2}}\right) + 1\right] = -\left(\lim_{h\to 0} (-h) + 1\right)$$

$$= -(-0 + 1) = -1$$

Thus we get, $\int_0^1 \log x\,dx = -1$

5. Show that $\int_0^{\infty} e^{-x}\, x^n\,dx = n!$, where n being a positive integer.

Solution:

$$\text{Let } I_n = \int_0^{\infty} e^{-x}\, x^n\,dx = \lim_{b\to\infty} \int_0^b e^{-x} x^n\,dx$$

$$= \lim_{b\to\infty} \left\{[-e^{-x} x^n]_0^b + n\int_0^b e^{-x} x^{n-1} dx\right\}$$

$$= n \lim_{b\to\infty} \int_0^b e^{-x} x^{n-1}\,dx, \quad \text{Since } \lim_{b\to\infty} e^{-b} b^n = 0$$

$$= n\,I_{n-1} = n(n-1)I_{n-2}$$

$$= n(n-1)(n-2)\,I_{n-3}$$

$$= n(n-1)(n-2)\ldots 2.1\, I_0$$

$$= n(n-1)(n-2)\ldots 2.1 \int_0^{\infty} e^{-x}\,dx$$

$$= n!\left(\int_0^{\infty} e^{-x} dx\right)$$

$$\text{Here, } \int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left(-e^{-x}\right)_0^b$$

$$= \lim_{b\to\infty} (1 - e^{-b})$$

$$= 1$$

Thus, we get, $\int_0^{\infty} e^{-x}\, x^n\,dx = n!$

6. Show that $\int_0^{\infty} e^{-ax} \sin bx\,dx = \frac{b}{a^2+b^2}$ for $a > 0$.

Solution:

Here, the improper integral $\int_0^{\infty} e^{-ax} \sin bx\,dx$ can be written as

$$I = \lim_{h\to\infty} \int_0^h e^{-ax} \sin bx\,dx$$

$$= \lim_{h\to\infty} \left[\frac{e^{-ax}}{a^2+b^2}(-a\sin bx - b\cos bx)\right]_0^h$$

$$= -\frac{1}{a^2+b^2} \lim_{h\to\infty} \left[e^{-ax}(a\sin bx + b\cos bx)\right]_0^h$$

$$= -\frac{1}{a^2+b^2} \lim_{h\to\infty} \left[e^{-ah}(a\sin bh + b\cos bh) - b\right]$$

$$= -\frac{1}{a^2+b^2}(0-b)$$

[Since $\lim\limits_{h\to\infty} e^{-ah} \to 0$, for $a > 0$ and sinbh and cosbh are bounded functions]

$$= \frac{b}{a^2+b^2}$$

Thus we get, $\displaystyle\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}$.

7. Evaluate the integral $\displaystyle\int_0^\infty \frac{\sin bx}{x}\, dx$

Solution:

$$\text{Let } I = \int_0^\infty \frac{e^{-ax}\sin bx}{x}\, dx, \text{ for } a > 0$$

Differentiate both sides with respect to b

$$\frac{dI}{db} = \int_0^\infty \frac{e^{-ax}\, x\cos bx}{x}\, dx$$

$$= \int_0^\infty e^{-ax} \cos bx\, dx$$

$$\therefore \quad \frac{dI}{db} = \frac{a}{a^2+b^2} \text{ for } a > 0$$

or, $\quad dI = \dfrac{a}{a^2+b^2}\, db$

Integrating both sides,

$$I = a \cdot \frac{1}{a}\tan^{-1}\frac{b}{a} + c$$

$$I = \tan^{-1}\left(\frac{b}{a}\right) + c \quad \text{............ (1)}$$

Where c is an integrating constant.

We have, from given integral, when $b = 0$, $I = 0$. Substituting these values in equation (1),

$$0 = \tan^{-1}(0) + c$$

or, $\quad c = 0$

Thus we get,

$$I = \tan^{-1}\left(\frac{b}{a}\right)$$

or, $\quad \displaystyle\int_0^\infty \frac{e^{-ax}\sin bx}{x}\, dx = \tan^{-1}\left(\frac{b}{a}\right)$

Taking limit $a \to 0$ on both sides,

$$\lim_{a\to 0}\int_0^\infty \frac{e^{-ax}\sin bx}{x}\, dx = \lim_{a\to 0}\tan^{-1}\left(\frac{b}{a}\right)$$

or, $\quad \displaystyle\int_0^\infty \frac{\sin bx}{x}\, dx = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$ for $b > 0$ or < 0 respectively.

Thus we get,

$$\int_0^\infty \frac{\sin bx}{x}\, dx = \begin{cases} \dfrac{\pi}{2} & \text{for } b > 0 \\ -\dfrac{\pi}{2} & \text{for } b < 0 \end{cases}$$

EXERCISE 11.7

Evaluate the following improper integral if it exists.

1. $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$
2. $\displaystyle\int_0^\infty \frac{x\, dx}{x^2+4}$
3. $\displaystyle\int_2^\infty \frac{x\, dx}{x^2-1}$
4. $\displaystyle\int_0^\infty x e^{-x^2}\, dx$

5. $\int_{-1}^{1} \frac{dx}{x^3}$

6. $\int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1}$

7. $\int_{1}^{\infty} \frac{\log x}{x^2}$

8. $\int_{1}^{\infty} \frac{x\,dx}{(1+x^2)^2}$

9. $\int_{0}^{1} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

10. $\int_{0}^{\pi} \frac{\sin x\,dx}{\cos^2 x}$

11. $\int_{0}^{\pi} \frac{dx}{1+\cos x}$

12. $\int_{0}^{a} \sqrt{\frac{a-x}{x}}\,dx$

13. $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,dx$

14. $\int_{0}^{2} \frac{dx}{(1-x)^2}$

15. $\int_{0}^{1} \log x\,dx$

16. $\int_{0}^{\infty} e^{-ax} \cos bx\,dx$ for $a > 0$

ANSWERS

1.	$\frac{\pi}{2}$	2.	Does not exists		
3.	~~$\frac{\log 3}{2}$~~ Does not exist.	4.	$\frac{1}{2}$	5.	0
6.	0	7.	1	8.	$\frac{1}{4}$
9.	$\frac{\pi^2}{8}$	10.	Does not exists		
11.	Does not exists	12	$\frac{a\pi}{2}$	13.	π
14.	Does not exists	15.	-1	16.	$\frac{a}{a^2+b^2}$

Integration by Successive reduction

In this chapter, the integrations are solved by successive reduction of the integrand which mostly depends on the repeated application of integration by parts. Most of the integrand of the types $x^n e^{ax}$, $\tan^n x$, $(x^2 + a^2)^{n/2}$, $\sin^m x$, $\cos^n x$ etc. These integrals are denoted by I_n or J_n. Reducing these integral whose suffixes are lower than that of the original integral until the last integral can be easily evaluated.

Example 1.

Find the reduction formula for $\int x^n e^{ax}\,dx$

Solution:

4, 8, 10, 11, 12, 13, 14,

$$\text{Let } I_n = \int x^n e^{ax}\,dx$$

$$= \frac{x^n e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a}\,dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

Which is the required reduction formula for

$$I_n = \int x^n e^{ax}\,dx$$

Example 2.

Find reduction formula for $\int \sin^n x\,dx$ and $\int_0^{\pi/2} \sin^n x\,dx$ and then evaluate $\int \sin^5 x\,dx$ and $\int_0^{\pi/2} \sin^5 x\,dx$

Solution:

$$\text{Let } I_n = \int \sin^n x\,dx$$

$$= \int \sin^{n-1} x \sin x\,dx$$

$$= \sin^{n-1}x(-\cos x) - (n-1)\int \sin^{n-2} x \cos x.(-\cos x)\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \cos^2 x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\left(\int \sin^{n-2}x\,dx - \int \sin^n x\,dx\right)$$

$$I_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2} - (n-1)I_n$$

or, $(1+n-1)I_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$

$$\therefore \quad I_n = -\frac{\cos x \sin^{n-1}x}{n} + \frac{n-1}{n}I_{n-2}$$

This is the required reduction formula for $I_n = \int \sin^n x\,dx$.

Again, let $J_n = \int_0^{\pi/2} \sin^n x\,dx$

$$= -\left(\frac{\sin^{n-1}x\cos x}{n}\right)_0^{\pi/2} + \frac{n-1}{n}J_{n-2}$$

$$J_n = \frac{n-1}{n}J_{n-2}$$

For $\int \sin^5 x\,dx = I_5$

$$= -\frac{\sin^4 x\cos x}{5} + \frac{5-1}{5}I_3$$

$$= -\frac{\sin^4 x\cos x}{5} + \frac{4}{5}\left[-\frac{\sin^2 x\cos x}{3} + \frac{2}{3}I_1\right]$$

Where $I_1 = \int \sin x\,dx = -\cos x$

Thus we get,

$$\int \sin^5 x\,dx = -\frac{\sin^4 x\cos x}{5} - \frac{4}{15}\sin^2 x\cos x + \frac{8}{15}(-\cos x) + c$$

$$= -\frac{\sin^4 x\cos x}{5} - \frac{4}{15}\sin^2 x\cos x - \frac{8}{15}\cos x + c$$

Again for $\int_0^{\pi/2} \sin^5 x\,dx = J_5 = \frac{5-1}{5}J_3$

$$= \frac{4}{5}\left[-\frac{3-1}{3}J_1\right] = \frac{8}{15}J_1$$

Where $I_1 = \int_0^{\pi/2} \sin x\,dx = (-\cos x)_0^{\pi/2}$

$$= \left(\cos\frac{\pi}{2} - \cos 0\right) = 1$$

Thus, we get

$$\int_0^{\pi/2} \sin^5 x\,dx = \frac{8}{15}$$

Example 3.

Find reduction formula for $\int \tan^n x\,dx$ and $\int_0^{\pi/4} \tan^n x\,dx$

Solution:

Let $I_n = \int \tan^n x\,dx$

$$= \int \tan^{n-2}x\,\tan^2 x\,dx$$

$$= \int \tan^{n-2}x(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2}x\sec^2 x\,dx - \int \tan^{n-2}x\,dx$$

$$\therefore \quad I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$$

This is the reduction formula for $I_n = \int \tan^n x\,dx$

Again let $J_n = \int_0^{\pi/4} \tan^n x\,dx$

$$= \left(\frac{\tan^{n-1}x}{n-1}\right)_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2}x\,dx$$

$$J_n = \frac{1}{n-1} - J_{n-2}$$

This is the required reduction formula for $J_n = \int_0^{\pi/4} \tan^n x\, dx$

Note: Putting different values of n, we can get the value of integral.

Example 4.

Find the formula for $\int \sec^n x\, dx$ and then evaluate $\int \sec^7 x\, dx$.

Solution:

$$\text{Let } I_n = \int \sec^n x\, dx$$

$$= \int \sec^{n-2} x \sec^2 x\, dx$$

$$= \sec^{n-2}x \tan x - (n-2) \int (\sec^{n-3}x \sec x \tan x \times \tan x dx)$$

$$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2} x (\tan^2 x)\, dx$$

$$= \sec^{n-2}x \tan x - (n-2) (\int \sec^n x\, dx - \int \sec^{n-2} x\, dx)$$

$$\therefore \quad I_n = \sec^{n-2}x \tan x - (n-2) I_n + (n-1) I_{n-2}$$

$$\text{or, } \quad (1+n-2)I_n = \sec^{n-2}x \tan x + (n-2)I_{n-2}$$

$$\therefore \quad I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

This is the required reduction formula for $\int \sec^n x\, dx$.

Note: If the integrals $\operatorname{cosec}^n x$, $\operatorname{sech}^n x$, $\operatorname{cosech}^n x$ then proceeding as above we can get the reduction formulas for these integrals.

$$\text{Again for } \int \sec^7 x\, dx = I_7 = \frac{\sec^5 x \tan x}{6} + \frac{5}{6} I_5$$

$$= \frac{\sec^5 x \tan x}{6} + \frac{5}{6}\left[\frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_1\right]$$

$$\text{Where } I_1 = \int \sec x\, dx$$

$$= \log(\sec x + \tan x)$$

Thus we get,

$$\int \sec^7 x\, dx = \frac{\sec^5 x \tan x}{6} + \frac{5}{24} \sec^3 x \tan x + \frac{15}{24} \log(\sec x + \tan x) + c$$

Example 5.

Find reduction formula for $\int \cos^m x \sin nx\, dx$ & $\int_0^{\pi/2} \cos^m x \sin nx\, dx$.

Solution:

$$\text{Let } I_{m,n} = \int \cos^m x \sin nx\, dx$$

$$= \cos^m x \left(\frac{\cos nx}{-n}\right) - \int m \cos^{m-1} x (-\sin x) \left(\frac{\cos nx}{-n}\right) dx$$

$$= -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n} \int \cos^{m-1} x (\sin x \cos nx) dx \quad \text{... (1)}$$

We have,

$$\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$$

$$\text{or, } \quad \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$$

From equation (1),

$$I_{m,n} = -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n} \int \cos^{m-1} x [\sin nx \cos x - \sin(n-1)x] dx$$

$$\text{or, } \quad I_{m,n} = -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n} \int \sin nx \cos^m x\, dx + \frac{m}{n} \int \cos^{m-1} x \sin(n-1)x\, dx$$

$$\text{or, } \quad I_{m,n} = -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\text{or, } \quad \left(1+\frac{m}{n}\right) I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\therefore \quad I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{n} I_{m-1,n-1}$$

This is the required reduction formula for

$$I_{m,n} = \int \cos^m x \sin nx\, dx$$

Again for the reduction formula for $\int_0^{\pi/2} \cos^m x \sin nx\, dx$

Let $J_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\, dx$

$$= \left(-\frac{\cos^m x \cos nx}{m+n}\right)_0^{\pi/2} + \frac{m}{m+n} J_{m-1,n-1}$$

$$J_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} J_{m-1,n-1}$$

This is the required reduction formula for $J_{m,n}$.

EXERCISE 11.8

1. Find a reduction formula for $\int \cos^n x\, dx$ and then evaluate $\int \cos^7 x\, dx$.

2. Find reduction formula for $\int_0^{\pi/2} \cos^n x\, dx$ and then evaluate $\int_0^{\pi/2} \cos^7 x\, dx$.

3. Find the reduction formula for $\int \cot^n x\, dx$ and then evaluate $\int \cot^7 x\, dx$.

4. Find the reduction formula for $\int \operatorname{cosec}^n x\, dx$ and then evaluate $\int \operatorname{cosec}^5 x\, dx$.

5. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, then show that $I_n = \frac{1}{n-1} - I_{n-2}$ and then find the value of I_5.

6. Find the reduction formula for $\int \cos^m x \cos nx\, dx$ and then show that $\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$

7. Find the reduction formula for $\int \cos^m x \sin nx\, dx$ and then evaluate $\int \cos^2 x \sin 3x\, dx$.

8. Show that $\int_0^{\pi/2} \cos^5 x \sin 3x\, dx = \frac{1}{3}$.

9. Show that $\int_0^{\pi/2} \cos^m x \sin mx\, dx = \frac{1}{2^{m+1}}\left[\frac{2^m}{m} + \frac{2^{m-1}}{m-1} + \ldots + \frac{2^2}{2} + 2\right]$

10. Find the reduction formula for $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\, dx$.

11. If $I_n = \int \sinh^n x\, dx$, then show that $nI_n = \sinh^{n-1} x \cosh x - (n-1)I_{n-2}$

12. Show that the reduction formula for the integral $I_n = \int (x^2 + a^2)^n\, dx$ is $I_n = \frac{x(x^2+a^2)^n}{2n+1} + \frac{2na^2}{2n+1} I_{n-1}$.

ANSWERS

1. $I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$ and

$I_7 = \frac{\cos^6 x \sin x}{7} + \frac{6}{35}\cos^4 x \sin x + \frac{24}{105}\cos^2 x \sin x + \frac{48}{105}\sin x + c$

2. $I_n = \frac{n-1}{n} I_{n-2}$ and $I_7 = \frac{16}{35}$

3. $I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$ and

$I_7 = -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} + \log(\sin x)$

4. $I_n = -\frac{\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

and $I_5 = -\frac{\operatorname{cosec}^3 x \cot x}{4} - \frac{3}{8}\operatorname{cosec} x \cot x + \frac{3}{8}\log\left(\tan\frac{x}{2}\right) + c$

5. $\frac{1}{2}\log 2 - \frac{1}{4}$

6. $I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$

7. $I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$

and $I_{2,3} = -\frac{\cos^2 x \cos 3x}{5} - \frac{2\cos x \cos 2x}{15} - \frac{2\cos x}{15}$

10. $I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$

Beta and Gamma Functions

Beta and gamma functions plays important role for applications of integral calculus. In this chapter we give their definitions, important properties and related problems.

Definition:

The integral of the form $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for $m > 0, n > 0$ is said to be first Eulerian integral or beta function and denoted by $\beta(m, n)$.

That is, $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for $m > 0, n > 0$.

Also the integral of the form $\int_0^\infty e^{-x}x^{n-1}\,dx$, for $n > 0$, is said to be second Eulerian integral or gamma function and denoted by Γn.

That is, $\Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx$, for $n > 0$

Note: In both cases m and n are positive but they need not be integers.

Properties:

1. $\beta(m, n) = \beta(n, m)$

Proof:

We know, $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $1 - x = t$, differentiating

$-dx = dt \quad \Rightarrow dx = -dt$

when $x = 0, t = 1$

when $x = 1, t = 0$

Then, $\beta(m, n) = \int_1^0 (1-t)^{m-1}t^{n-1}(-dt)$

$= \int_0^1 t^{n-1}(1-t)^{m-1}\,dt$

$\beta(m, n) = \beta(n, m)$

2. $\Gamma 1 = 1$

Proof:

We have $\Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx$

Put $n = 1$,

$\Gamma 1 = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$

$= \lim_{b\to\infty}\left(\frac{e^{-x}}{-1}\right)_0^b = \lim_{b\to\infty}\left(\frac{e^{-x}-1}{-1}\right)$

$= \lim_{b\to\infty}(1 - e^{-b})$

$= 1 \quad$ [Since $\lim_{b\to\infty} e^{-b} = 0$]

Thus we get, $\Gamma 1 = 1$

3. $\Gamma(n+1) = n\Gamma(n)$

Proof:

We have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

then $\Gamma(n+1) = \int_0^\infty e^{-x} x^{n}\, dx = \lim_{b \to \infty} \int_0^b e^{-x} x^n\, dx$

$$= \lim_{b \to \infty} \left[\left(\frac{x^n e^{-x}}{-1} \right)_0^b - \int_0^b n x^{n-1} \frac{e^{-x}}{-1} dx \right]$$

$$= \lim_{b \to \infty} (-b^n e^{-b}) + n \int_0^\infty x^{n-1} e^{-x}\, dx$$

$$= 0 + n \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = n\Gamma(n)$$

Note:

i) We have,

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n(n-1)\,\Gamma(n-1)$$
$$= n(n-1)(n-2)\,\Gamma(n-2)$$
$$= n(n-1)(n-2) \ldots\ldots. 3.\,2.\,1\,\Gamma(1)$$
$$= n(n-1)(n-2) \ldots 3.\,2.\,1$$
$$\Gamma(n+1) = n!$$

ii) Writing kx for x in $\Gamma(n)$, then we get $\int_0^\infty e^{-kx} x^{n-1}\, dx = \frac{\Gamma(n)}{k^n}$

(for $k > 0$, $n > 0$)

4. $\beta(m, n) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$

Proof:

We have, $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

$$= \left[(1-x)^{n-1} \frac{x^m}{m} \right]_0^1 + (n-1) \int_0^1 (1-x)^{n-2} \frac{x^m}{m} dx$$

$$= \frac{n-1}{m} \int_0^1 (1-x)^{n-2} x^m\, dx$$

Again integrating,

$$= \frac{n-1}{m} \frac{n-2}{m+1} \int_0^1 (1-x)^{n-3} x^{m+1}\, dx$$

Proceeding in this way, we get

$$\beta(m, n) = \frac{(n-1)}{m} \frac{n-2}{m+1} \frac{(n-3)\ldots2.1}{(m+2)\ldots(m+n-2)} \int_0^1 (1-x)^{n-n}\; x^{m+n-2}\, dx$$

$$= \frac{(n-1)!}{m(m+1)\ldots(m+n-2)} \left(\frac{x^{m+n-1}}{m+n-1} \right)_0^1$$

$$= \frac{(n-1)!}{m(m+1)\ldots(m+n-2)(m+n-1)}$$

$$= \frac{(n-1)!\,1.2.3\ldots(m-1)}{1.2.3\ldots(m-1)\; m(m+1)\ldots(m+n-1)}$$

$$= \frac{(n-1)!\,(m-1)!}{(m+n-1)!} = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$$

Thus we get, $\beta(m, n) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$

5. $\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

Proof:

We have, $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Put $x = \frac{1}{1+y} \Rightarrow dx = \frac{-dy}{(1+y)^2}$

When $x = 0,\ y \to \infty$ and when $x = 1,\ y = 0$

Then, we get,

$$\beta(m, n) = \int_{\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1} \left(1-\frac{1}{1+y}\right)^{n-1} \left[-\frac{dy}{(1+y)^2}\right]$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+1}(1+y)^{n-1}} dy$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\beta(m, n) = \int_0^{\infty} \frac{x^{n-1}dx}{(1+x)^{m+n}} = \beta(n, m)$$

$$= \int_0^{\infty} \frac{x^{m-1}dx}{(1+x)^{n+m}}$$

Thus, we get, $\beta(m, n) = \int_0^{\infty} \frac{x^{n-1}dx}{(1+x)^{n+m}} = \int_0^{\infty} \frac{x^{m-1}dx}{(1+x)^{m+n}}$

6. $\Gamma m \Gamma 1-m = \frac{\pi}{\sin m\pi}$ for $0 < m < 1$.

Proof:

Here, $\Gamma m \Gamma 1-m = \frac{\Gamma m \Gamma 1-m}{\Gamma m+1-m} = \beta(m, 1-m)$

$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+1-m}} dx$$

$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)} dx$$

$$= \frac{\pi}{\sin m\pi} \quad \left[\text{Since} \int_0^{\infty} \frac{x^{n-1}}{(1+x)} dx = \frac{\pi}{\sin n\pi}\right]$$

Thus we get, $\Gamma m \Gamma 1-m = \frac{\pi}{\sin m\pi}$

Note:

i) Put $m = \frac{1}{2}$, then we get $\Gamma \frac{1}{2} \Gamma 1-\frac{1}{2} = \frac{\pi}{\sin \frac{\pi}{2}}$

$$\Rightarrow \Gamma \tfrac{1}{2} \Gamma \tfrac{1}{2} = \pi$$

$$\Rightarrow \left(\Gamma \tfrac{1}{2}\right)^2 = \pi$$

$$\therefore \Gamma \tfrac{1}{2} = \sqrt{\pi}$$

ii) $\Gamma \frac{1}{2} = \sqrt{\pi}$ can be derived by using this formula $\frac{\Gamma m \Gamma n}{\Gamma m+n} = \beta(m, n)$ by putting $m = \Gamma \frac{1}{2} = n$.

7. $$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma \frac{m+1}{2} \Gamma \frac{n+1}{2}}{2\Gamma \frac{m+n+2}{2}}$$

Proof:

Here, $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$

$$= \int_0^{\pi/2} (\sin^2 x)^{m/2}(1-\sin^2 x)^{n/2} dx$$

Put $\sin^2 x = t$, differentiating,

$2\sin x \cos x\, dx = dt$

$dx = \frac{dt}{2\sqrt{t}\sqrt{1-t}}$

When $x = 0,\ t = 0$

$x = \frac{\pi}{2},\ t = 1$, then

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \int_0^1 t^{m/2}(1-t)^{n/2} \frac{dt}{2\sqrt{t}\sqrt{1-t}}$$

$$= \frac{1}{2}\int_0^1 t^{\frac{m}{2}-\frac{1}{2}}(1-t)^{\frac{n}{2}-\frac{1}{2}} dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{m+1}{2}-1}(1-t)^{\frac{n+1}{2}-1} dt$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\frac{m+1}{2}\Gamma\frac{n+1}{2}}{\Gamma\left(\frac{m+1}{2}+\frac{n+1}{2}\right)} = \frac{\Gamma\frac{m+1}{2}\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$$

Thus we get,

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\frac{m+1}{2}\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$$

Note:

i) When $n = 0$, we get

$$\int_0^{\pi/2} \sin^m x \, dx = \frac{\Gamma\frac{m+1}{2}\Gamma\frac{1}{2}}{2\Gamma\frac{m+2}{2}} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\frac{m+1}{2}}{\Gamma\frac{m+2}{2}}$$

ii) When $m = 0$, we get

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{\Gamma\frac{1}{2}\Gamma\frac{n+1}{2}}{2\Gamma\frac{n+2}{2}} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}}$$

8.

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Proof:

For $\int_0^{\infty} e^{-x^2} dx$

Put $x^2 = t$, differentiating, $2x \, dx = dt$

$$dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$$

When $x = 0, t = 0$

When, $x \to \infty, t \to \infty$, then

$$\int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-t}\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^{\infty} e^{-t} t^{-1/2} dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{2}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$$

Thus we get, $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Examples

1. Evaluate (i) $\Gamma 5$ (ii) $\Gamma\frac{7}{2}$ (iii) $\frac{\Gamma\frac{5}{2}\Gamma\frac{7}{2}}{\Gamma 4}$

Solution:

(i) We have, $\Gamma n = (n-1)!$, then

$$\Gamma 5 = (5-1)! = 4! = 4.3.2.1 = 24$$

(ii) We have $\Gamma n+1 = n\Gamma n$, then

$$\Gamma\frac{7}{2} = \left(\frac{7}{2}-1\right)\Gamma\frac{7}{2}-1$$

$$= \frac{5}{2}\Gamma\frac{5}{2} = \frac{5}{2}.\frac{3}{2}\Gamma\frac{3}{2}$$

$$= \frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}$$

$$= \frac{5.3.1}{8}\Gamma\frac{1}{2} = \frac{15}{8}\sqrt{\pi}$$

(iii) Here, $\frac{\Gamma\frac{5}{2}\Gamma\frac{7}{2}}{\Gamma 4} = \frac{\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}{3!} = \frac{45\pi}{64\times 3} = \frac{15}{64}\pi$

2. Evaluate $\int_0^{\pi/2} \sin^3\theta\cos^4\theta\, d\theta$

Solution:

We have $\int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$

Here, $\int_0^{\pi/2} \sin^3\theta\cos^4\theta\, d\theta = \frac{\Gamma\frac{3+1}{2}\Gamma\frac{4+1}{2}}{2\Gamma\frac{3+4+2}{2}}$

$$= \frac{\Gamma 2\,\Gamma\frac{5}{2}}{2\Gamma\frac{9}{2}} = \frac{1!\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}} = \frac{1}{\frac{35}{4}} = \frac{4}{35}$$

Thus we get, $\int_0^{\pi/2} \sin^3\theta\cos^4\theta\, d\theta = \frac{4}{35}$.

3. Using gamma function evaluate $\int_0^1 x^{3/2}(1-x)^{3/2}\, dx$

Solution:

Here, $\int_0^1 x^{3/2}(1-x)^{3/2}\, dx$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

when $x = 0, \theta = 0$

$x = 1, \theta = \frac{\pi}{2}$, then

$$\int_0^1 x^{3/2}(1-x)^{3/2}\, dx = \int_0^1 (\sin^2\theta)^{3/2}(1-\sin^2\theta)^{3/2} . 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^3\theta\cos^3\theta\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^4\theta\cos^4\theta\, d\theta$$

$$= 2\frac{\Gamma\frac{4+1}{2}\Gamma\frac{4+1}{2}}{2\Gamma\frac{4+4+2}{2}} = 2\frac{\Gamma\frac{5}{2}\Gamma\frac{5}{2}}{2\Gamma\frac{10}{2}}$$

$$= \frac{\left(\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\right)^2}{4!} = \frac{9\pi}{4.4.3.2.1}$$

$$= \frac{3\pi}{32}$$

4. Using gamma function, evaluate $\int_0^{\pi} \sin^6\frac{x}{2}\cos^8\frac{x}{2}\, dx$

Solution:

The given integral is

$$I = \int_0^{\pi} \sin^6\frac{x}{2}\cos^8\frac{x}{2}\, dx$$

Put $\frac{x}{2} = \theta \quad \Rightarrow x = 2\theta$, differentiating,

$dx = 2\, d\theta$

when $x = 0, \theta = 0$

$x = \pi, \theta = \frac{\pi}{2}$, then

$$I = \int_0^{\pi} \sin^6\frac{x}{2}\cos^8\frac{x}{2}\, dx$$

$$= \int_0^{\pi} \sin^6\theta\cos^8\theta\, 2\, d\theta$$

$$= 2\frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{8+1}{2}\right)}{2\Gamma\left(\frac{6+8+2}{2}\right)} = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{9}{2}\right)}{\Gamma(8)}$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{7!}$$

$$= \frac{5\times3\times7\times5\times3\sqrt{\pi}\times\sqrt{\pi}}{128\times7\times6\times5\times4\times3\times2\times1} = \frac{5\pi}{128\times4} = \frac{5\pi}{512}$$

5. Evaluate $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta$

Solution:

Here, given integral $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta \, d\theta$

Put, $3\theta =$ x, differentiating,

$3d\theta = dx$

when $\theta = 0, x = 0$

when $\theta = \frac{\pi}{6}, x = \frac{\pi}{2}$, then

$$I = \int_0^{\pi/2} \cos^4 x \sin^2 2x \frac{dx}{3}$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^4 x(2\sin x\cos x)^2 \, dx$$

$$= \frac{4}{3}\int_0^{\pi/2} \cos^6 x \sin^2 x \, dx$$

$$= \frac{4}{3}\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(\frac{10}{2}\right)}$$

$$= \frac{2}{3}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{7}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2\times4!} = \frac{5\pi}{192}$$

6. Show that $\frac{\beta(m, n+1)}{n} = \frac{\beta(m+1, n)}{m} = \frac{\beta(m, n)}{m+n}$

Proof:

$$\frac{\beta(m, n+1)}{n} = \frac{\Gamma m\,\Gamma n+1}{n\,\Gamma m+n+1}$$

$$= \frac{\Gamma m\, n\Gamma n}{n(m+n)\Gamma m+n} = \frac{\Gamma m\,\Gamma n}{(m+n)\Gamma m+n}$$

$$= \frac{1}{m+n}\frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \frac{1}{m+n}\beta(m, n)$$

Similarly we get, $\frac{\beta(m+1, n)}{m} = \frac{\beta(m, n)}{m+n}$

7. Evaluate $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx$

Solution:

Here, given integral

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx$$

$$= \int_0^{\pi/2} (\sin x)^{-1/2} dx \times \int_0^{\pi/2} (\sin x)^{1/2} dx$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)} \times \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+2}{2}\right)} = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}\pi$$

$$= \frac{\pi}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)} = \pi\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}$$

$$= \pi$$

Thus we get, $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx = \pi$

EXERCISE 11.9

1. Evaluate followings:

   i) $\Gamma(9)$ ii) $\Gamma\left(\frac{11}{2}\right)$ iii) $\dfrac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(7)}$

2. Evaluate i) $\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx$ ii) $\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx$

3. Evaluate following integrals using beta and gamma functions:

   i) $\int_0^1 \dfrac{x^6 dx}{\sqrt{1-x^2}}$ [Hint: Put $x = \sin\theta$]

   ii) $\int_0^{2a} x^5\sqrt{(2a-x)x}\, dx$ [Hint: Put $x = 2a\sin^2\theta$]

   iii) $\int_0^a x^4\sqrt{a^2-x^2}\, dx$ [Hint: Put $x = a\sin\theta$]

   iv) $\int_0^a x^2(a^2-x^2)^{3/2}\, dx$ [Hint: Put $x = a\sin\theta$]

   v) $\int_0^{2a} x^{9/2}\sqrt{(2a-x)}\, dx$ [Hint: Put $x = 2a\sin^2\theta$]

   vi) $\int_0^1 x^6\sqrt{1-x^2}\, dx$ [Hint: Put $x = \sin\theta$]

   vii) $\int_0^a \dfrac{x^4 dx}{\sqrt{a^2-x^2}}$ [Hint: Put $x = a\sin\theta$]

   viii) $\int_0^1 \dfrac{dx}{(1-x^6)^{1/6}}$ [Hint: Put $x^6 = y$ / $x^3 = \sin\theta$]

4. Evaluate following integrals by using beta and gamma functions:

   i) $\int_0^{\pi/8} \cos^3 4x\, dx$ [Hint: Put $4x = t$]

   ii) $\int_0^{\pi/4} \sin^4 x\cos^2 x\, dx$ [Hint: First transform to 2x form and then put $2x = t$]

   iii) $\int_0^{\pi/6} \cos^2 6\theta \sin^4 3\theta\, d\theta$ [Hint: Put $3\theta = t$]

   iv) $\int_0^{\pi/4} (1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta$ [Hint: Put $\sqrt{2}\sin\theta = \sin x$]

5. Evaluate $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

6. Show that $\beta(n, m)\beta(m+n, l) = \beta(m, l)\beta(m+l, n) = \beta(l, n)\,\beta(l+n, m)$.

7. Show that $\Gamma\left(\frac{2n+1}{2}\right) = [1.\,3.\,5 \ldots\ldots (2n-3)\,(2n-1)]\,\dfrac{\sqrt{\pi}}{2^n}$

8. Show that $\int_{-1}^1 (1+x)^p(1-x)^q\, dx = 2^{p+q+1}\dfrac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$ for $p > -1, q > -1$.

   [Hint: Put $1 + x = 2y$]

9. Show that $\int_b^a (x-a)^m(b-x)^n\, dx = (b-a)^{m+n+1}\dfrac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$ for $m > -1$, $n > -1$. [Hint: $x - a = (b-a)y$]

10. Show that $\int_0^\infty e^{-x^2}x^\lambda\, dx = \dfrac{1}{2}\Gamma\left(\dfrac{\lambda+1}{2}\right)$ for $\lambda > -1$.

    [Hint: Put $x^2 = y$]

11. Show that $\int_0^\infty e^{-x^4}x^2\, dx \times \int_0^\infty e^{-x^4}\, dx = \dfrac{\pi}{8\sqrt{2}}$ [Hint: Put $x^4 = y$]

12. Show that $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right) = \frac{3\pi}{16}$

[Hint: Combine first and last factor, second and last but one etc. and apply the formula $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$, for $0 < m > 1$.]

ANSWERS

1. i) $8! = 40320$ ii) $\frac{945}{32}\sqrt{\pi}$ iii) $\frac{7\pi}{1024}$
2. i) $\frac{2}{35}$ ii) $\frac{5\pi}{4096}$
3. i) $\frac{5\pi}{32}$ ii) $\frac{33}{16}\pi a^7$ iii) $\frac{\pi a^6}{32}$
   iv) $\frac{\pi a^6}{32}$ v) $\frac{63}{8}\pi a^5$ vi) $\frac{5\pi}{256}$
   vii) $\frac{3\pi a^4}{16}$ viii) $\frac{\pi}{3}$
4. i) $\frac{1}{6}$ ii) $\frac{3\pi - 4}{192}$ iii) $\frac{7\pi}{192}$
   iv) $\frac{3\pi}{16\sqrt{2}}$
5. $\pi\sqrt{2}$

Definite integral as a Limit of a sum

Let f(x) be a function continuous in the interval [a,b] where $b > a$. Let the interval [a, b] be divided into n equal parts each having length h, then $nh = b - a$.

We define

$$S = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+\ldots\ldots\ldots\ldots +f\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a + rh)$$

as the definite integral of f(x) with respect to x between the limits a and b provided the limit exists.

The above expression is written in short as. $\int_a^b f(x)\,dx$.

That is, $$\int_a^b f(x)dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh), \quad \text{where } nh = b - a.$$

OR

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=0}^{n-1} f\left[a+(b-a)\frac{r}{n}\right]$$

Since each term of the series $h\Sigma f(a+rh)$ tends to zero, we may add or omit the terms hf(a) and hf(a+nh). So one may write

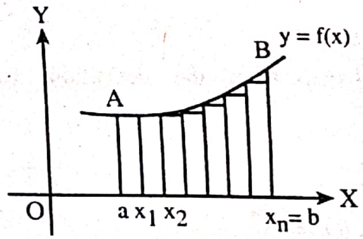
$$\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$$

OR

$$= \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh) \ldots\ldots (i)$$

Geometrically

Let us consider a function $y = f(x)$ continuous and non negative in the interval [a,b]. Let AB represents the curve of $y = f(x)$ in [a, b].



Now,

We divide the interval [a,b] into n equal parts such that

$a = x_0$ and $b = x_n$, where, $b - a = nh$

The ordinates at

$x_0, x_1, x_2, \ldots, x_n$ are f(a), f(a+h), f(a+2h), ..., f[a+(n−1)h]

Completing the rectangles as shown in the figure, the areas of the inner rectangles are

hf(a), hf(a+h), hf(a+2h) ......, hf[a+(n−1)h]

∴ Sum of the areas of inner rectangles

= hf(a) + hf(a+h) + hf(a+2h)+...+ hf[a+(n−1)h] —(i)

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Again, the heights of the outer rectangles are the lengths of the ordinates at $x_1, x_2, \ldots, x_n$.

$\therefore$ The sum of the areas of outer rectangles

$= hf(a+h)+hf(a+2h)+\ldots\ldots+hf(a+nh)$ ......(ii)

But the area Ax_0x_nB lies between the two areas (i) and (ii),

i.e. $h\sum_{r=0}^{n-1} f(a+rh) <$ area $Ax_0x_nB < h\sum_{r=0}^{n-1} f(a+rh)$

In the limit when $n\to\infty$, $h\to 0$ and we have defined

$$\lim_{h\to 0} \sum_{r=0}^{n-1} f(a+rh)$$

or, $$\lim_{h\to 0} \sum_{r=0}^{n-1} f(a+rh)$$

$$\therefore \int_a^b f(x)\, dx = \text{Area } Ax_0x_nB$$

which is the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$.

Example 1.

Evaluate $\int_0^1 x^2\, dx$ from the definition. (or ab-initio) (or by summation method)

Solution:

Here, $f(x) = x^2$

$\therefore f(a+rh) = (a+rh)^2$

Also, $a = 0$ and $b = 1$.

$\therefore nh = b-a = 1-0 = 1$

$\therefore f(a+rh) = f(a+rh) = (a+rh)^2 = r^2h^2$ $\qquad \because a = 0$

Now, from definition

$$\int_0^1 x^2\, dx = \lim_{h\to 0} h\sum_{r=1}^{n} r^2h^2$$

$$= \lim_{h\to 0} h^3 \sum_{r=1}^{n-1} r^2$$

$$= \lim_{h\to 0} h^3[1^2+2^2+3^2+\ldots\ldots+n^2]$$

$$= \lim_{h\to 0} h^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{h\to 0} \frac{nh(nh+h)(2nh+h)}{6}$$

$$= \lim_{h\to 0} \frac{1(1+h)(2\times 1+h)}{6}$$

$$= \lim_{h\to 0} \frac{(1+h)(2+h)}{6} = \frac{1}{3}.$$

Example 2.

Evaluate $\int_a^b e^{mx} dx$ as the limit of a sum.

Solution:

Here, $f(x) = e^{mx}$

$\therefore f(a+rh) = e^{m(a+rh)}$

Also, $a = a$ & $b = b$, so $nh = b-a$

Now,

$$\int_a^b e^{mx}\, dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} e^{m(a+rh)}$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} e^{ma}.e^{mrh}$$

$$= e^{ma} \lim_{h\to 0} h\sum_{r=1}^{n} e^{mrh}$$

$$= e^{ma} \lim_{h\to 0} h\left[e^{mh} + e^{2mh} + e^{3mh} + \ldots + e^{nmh}\right]$$

$$= e^{ma} \lim_{h\to 0} h\, e^{mh}\left[\frac{(e^{mh})^n - 1}{e^{mh}-1}\right]$$

$$= e^{ma} \lim_{h\to 0} e^{mh}(e^{mnh}-1) \lim_{h\to 0}\frac{mh}{e^{mh}-1}\times\frac{1}{m}$$

$$= e^{ma} \lim_{h\to 0} e^{mh}.\left\{\frac{e^{m(b-a)}-1}{m}\right\}, \text{ where } \lim_{h\to 0}\frac{mh}{e^{mh}-1} = 1$$

$$= e^{ma} e^{0}.\frac{e^{mb-ma}-1}{m}$$

$$= \frac{1}{m}\left[e^{mb-ma+ma} - e^{ma}\right]$$

$$= \frac{1}{m}\left[e^{mb} - e^{ma}\right]$$

Example 3.

Show that: $\int_a^b \sin x\, dx = \cos a - \cos b$, by ab-initio.

Solution:

We have

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh) \quad \ldots\ldots\ldots\ldots(1)$$

Here,

$$f(x) = \sin x$$
$$f(a+rh) = \sin(a+rh)$$

Then,

$$\int_a^b \sin x\,dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} \sin(a+rh)$$

$$= \lim_{h\to 0} h\,[\sin a + \sin(a+h) + \sin(a+2h) + \ldots + \sin(a+(n-1)h)]$$

$$= \lim_{h\to 0} h\,\frac{\sin\frac{nh}{2}\sin\left(a+\frac{(n-1)h}{2}\right)}{\left(\sin\frac{h}{2}\right)}$$

$$= \lim_{h\to 0} \frac{h}{\sin\frac{h}{2}}\sin\left(\frac{b-a}{2}\right)\sin\left(a+\frac{nh-h}{2}\right)$$

$$= \sin\left(\frac{b-a}{2}\right)\sin\left(a+\frac{b-a-0}{2}\right)\lim_{h\to 0}\left(\frac{h}{\sin\frac{h}{2}}\right)$$

$$= 2\left[\sin\left(\frac{b-a}{2}\right)\sin\left(\frac{b+a}{2}\right)\right]\lim_{h\to 0}\frac{\frac{h}{2}}{\sin\frac{h}{2}}$$

$$= 2\sin\frac{b-a}{2}\sin\left(\frac{a+b}{2}\right) = \cos a - \cos b$$

Thus we get, $\int_a^b \sin x\,dx = \cos a - \cos b$

Note:

The value of $\sum_{r=0}^{n-1} \cos(a+rh)$

$$= \cos a + \cos(a+h) + \ldots + \cos[a+(n-1)h]$$

$$= \frac{\sin\frac{nh}{2}\cos\left[\left(a+\left(\frac{n-1}{2}\right)h\right)\right]}{\sin\frac{h}{2}}$$

Example 4.

Evaluate $\int_a^b x^m dx$, for $m \neq -1$, is any rational number (positive or negative, by using as initio method.

Solution:

Let us divide the interval [a, b] in finite set of points a, ar, ar^2, …, ar^n (= b).

Where, $ar^n = b \Rightarrow r^n = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n}}$. When n tends to infinity, r tends to 1.

Then, $\int_a^b x^m dx = [f(a)(ar-a) + f(ar)(ar^2-ar) + f(ar^2)(ar^3-ar^2) + \ldots \text{ up to } n-\text{terms}]$

$$= \lim_{n\to\infty} [a(r-1)f(a) + ar(r-1)f(ar) + \ldots]$$

$$= \lim_{n\to\infty} a(r-1)(a^m + r(ar)m + \ldots + n-\text{term})$$

$$= \lim_{n\to\infty} a(r-1)a^m(1 + r^{m+1} + r^{2(m+1)} + \ldots)$$

$$= \lim_{r\to 1} a^{m+1}(r-1)\left[1 + r^{m+1} + \left(r^{m+1}\right)^2 + \ldots\right]$$

$$= a^{m+1} \lim_{r\to 1} (r-1)\left(\frac{(r^{m+1})^n - 1}{r^{m+1}-1}\right)$$

$$= a^{m+1} \lim_{r\to 1} \left(\frac{r-1}{r^{m+1}-1}\right)\left[\left(r^n\right)^{m+1} - 1\right]$$

$$= a^{m+1}\left[\left(\frac{b}{a}\right)^{m+1} - 1\right] \lim_{r\to 1}\left(\frac{r-1}{r^{m+1}-1}\right)$$

$$= (b^{m+1} - a^{m+1}) \lim_{r\to 1}\left[\frac{1}{(m+1)r^m}\right]$$

By using L'Hopital theorem.

$$= \frac{\left(b^{m+1} - a^{m+1}\right)}{m+1}$$

We get,

$$\therefore \int_a^b x^m dx = \frac{b^{m+1} - a^{m+1}}{m+1}.$$

Note: To integrate any power of x (except $\frac{1}{x}$), we first evaluate $\int_a^b x^m dx$ and then substitute the given value of m. For example, for $\int_a^b \sqrt{x}\, dx$; first evaluate,

$$\int_a^b x^m dx = \frac{b^{m+1} - a^{m+1}}{m+1} \text{ and then}$$

Put, $m = \frac{1}{2}$, then $\int_a^b x^m dx = \int_a^b x^{\frac{1}{2}} dx$

$$= \frac{b^{\frac{1}{2}+1} - a^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{b^{\frac{3}{2}} - a^{\frac{3}{2}}}{\frac{3}{2}}$$

Example 5.

Evaluate: $\int_a^b \frac{dx}{x}$, by using definition.

Solution:

Let us divide the internal [a, b] in to n-subintervals, by finite set of points a, ar, ar^2, … ar^n (=b). where $r = \left(\frac{b}{a}\right)^{\frac{1}{n}}$, which tends to unity, when n tends to infinity.

Then the integral

$$\int_a^b \frac{1}{x} dx = \lim_{n\to\infty} (f(a)(ar-a) + f(ar)(ar^2 - ar) + f(ar^2)(ar^3 - ar^2) + \ldots + n \text{ terms})$$

$$= \lim_{n\to\infty}\left(\frac{1}{a}a(r-1) + \frac{1}{ar}ar(r-1) + \frac{1}{ar^2}ar^2(r-1) + \ldots\right)$$

$$= \lim_{n\to\infty} [(r-1) + (r-1) + (r-1) + \ldots]$$

$$= \lim_{n\to\infty} n(r-1) \quad \ldots\ldots \text{(i)}$$

We have, $\log r = \frac{1}{n}\log\left(\frac{b}{a}\right)$, $\quad n = \frac{\log b/a}{\log r}$

From equation (i),

$$\int_a^b \frac{1}{x} dx = \lim_{r\to 1} \frac{\log b/a}{\log r}(r-1) \qquad [0/0 \text{ form}]$$

$$= \lim_{r\to 1} \frac{\log b/a}{1/r} = \log b/a$$

$$\therefore \int_a^b \frac{1}{x} dx = \log b - \log a.$$

EXERCISE 11.10

Evaluate the following integrals by the method of summation (i.e. by definition)

1. $\int_1^2 x^2 dx$
2. $\int_a^b e^{-x} dx$
3. $\int_1^4 (x^2 - x) dx$
4. $\int_0^1 \sqrt{x}\, dx$
5. $\int_0^1 (ax+b)\, dx$
6. $\int_a^b \cos x\, dx$
7. $\int_0^{\frac{\pi}{2}} \cos x\, dx$
8. $\int_0^{\frac{\pi}{2}} \sin x\, dx$
9. $\int_a^b \frac{dx}{x^2}$
10. $\int_a^b \frac{dx}{\sqrt{x}}$
11. $\int_a^b \sqrt{x}\, dx$
12. $\int_0^{\frac{\pi}{4}} \sin^2 x\, dx$
13. $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

ANSWERS

1. $\frac{7}{3}$
2. $e^{-a} - e^{-b}$
3. $\frac{27}{2}$
4. $\frac{2}{3}$
5. $\frac{a}{2} + b$
6) $\sin b - \sin a$
7) 1
8) 1
9) $\frac{b-a}{ab}$
10) $2((\sqrt{b} - \sqrt{a})$
11) $\frac{2}{3}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$
12) $\frac{\pi}{8} - \frac{1}{4}$
13) $\frac{\pi}{8} + \frac{1}{4}$

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