Mechanical Oscillation (SHM):

- Introduction and equation of SHM
- Energy in SHM
- Oscillation of mass-spring system
- Compound Pendulum

Oscillation:

- Motions that repeat themselves are called oscillations .
- They occur almost everywhere around us.

Examples:

- Oscillating guitar strings, drums
- bells, diaphragms in telephones and speaker systems, quartz crystals in wristwatches
- Oscillations of the air molecules that transmit the sensation of temperature
- Oscillations of the electrons in the antennas of radio and TV transmitters that convey information. (Source: Fundamentals of Physics :David Halliday, Robert Resnick, Jearl Walker)

Simple Harmonic Motion (SHM) :

Definition:

The motion in which the restoring force(F) is directly proportional to the displacement(x) from the mean position and is opposite to it is called Simple Harmonic Motion.

i.e. $F \propto x$

Or, $F = -kx$, where k is a constant called force constant and the negative sign is due to the opposite direction of F and x.

Definition of k:

- Since F = - kx,

k = $\frac{F}{X}$, in magnitude

So, the force constant is defined as the restoring force per unit displacement from the mean position.

Unit of k:

k = $\frac{F}{X}$ → $\frac{N}{m}$ (In SI System) or $\frac{dyne}{cm}$ (In CGS System)

Differential Equation of SHM:

Since $F = -kx$ and $F = ma$,

$ma = -kx$

Now, a can be written as $a = \frac{d^2x}{dt^2}$

So, $m\frac{d^2x}{dt^2} = -kx$

Or, $\frac{d^2x}{dt^2} + \omega^2 x = 0$, which is the required differential equation of SHM.

Contd…

The solution of the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = o$$

can be written as $$x = x_m \cos(\omega t + \phi)$$

The range of x lies between $-x_m$ to $+x_m$

Characteristics of SHM:

1. Displacement: It is given by

$$x = x_m \cos(\omega t + \phi) \tag{1}$$

2. Velocity: It is given by

$$v = \frac{dx}{dt}$$

$$= \frac{d[x_m \cos(\omega t + \phi)]}{dt}$$

$$= -\omega x_m \sin(\omega t + \phi) \tag{2}$$

Again, squaring (2), we get

$$v^2 = \omega^2 x_m^2 \sin^2(\omega t + \phi)$$
$$= \omega^2 x_m^2 [1 - \cos^2(\omega t + \phi)]$$
$$= \omega^2 x_m^2 - \omega^2 x_m^2 \cos^2(\omega t + \phi)$$
$$= \omega^2 [x_m^2 - x^2]$$

So,

$$\boxed{v = \pm\omega\sqrt{x_m^2 - x^2}} \qquad (3)$$

where $\omega = \sqrt{\dfrac{k}{m}}$ is the angular frequency

3. Acceleration:

It is given by

$$a = \frac{dv}{dt}$$

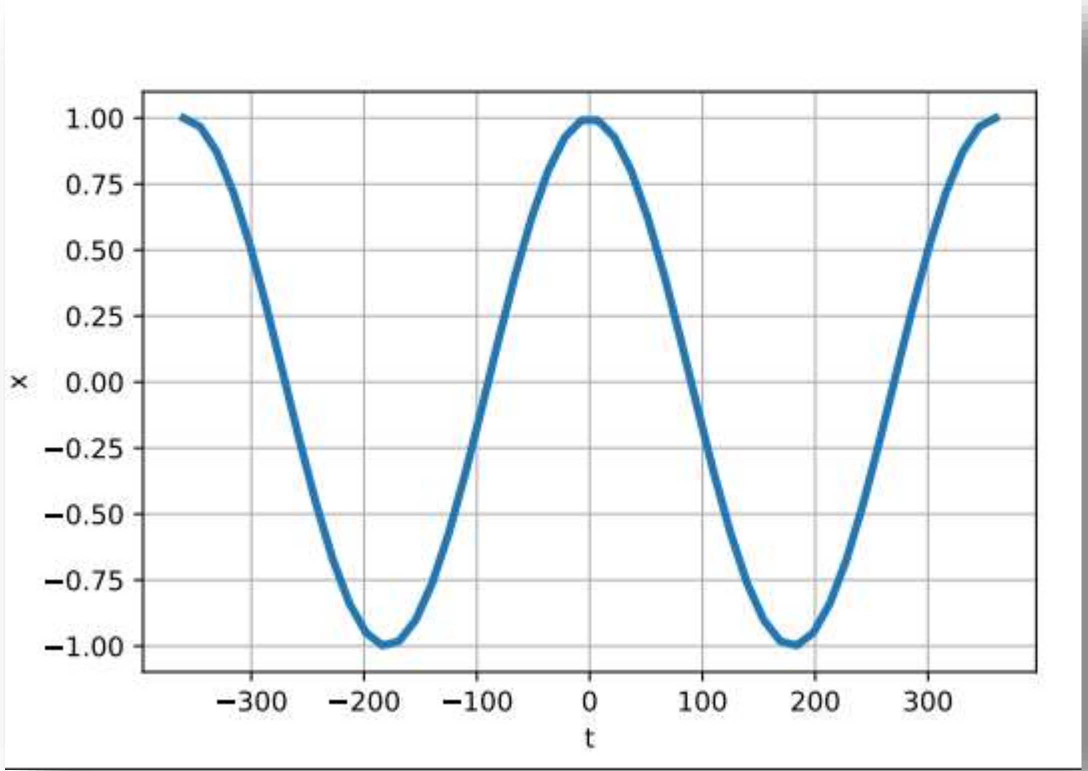
$$= \frac{d[-\omega x_m \sin(\omega t + \phi)]}{dt}$$

$$= -\omega x_m \cos(\omega t + \phi).\omega$$

$$= -\omega^2 x_m \cos(\omega t + \phi) \qquad (4)$$

But $x = x_m \cos(\omega t + \phi)$ (from eq. (1))

So, $\boxed{a = -\omega^2 x}$ $\qquad (5)$


1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
x
−300
−200
−100
0
100
200
300
t

4. Time period (T) :

It is the time taken by the body to complete one oscillation and is given by

$$T = \frac{2\pi}{\omega}$$

But $a = -\omega^2 x$ (From eqn (5))

So, $\omega^2 = \frac{a}{x}$ (In magnitude)

$$\boxed{T = 2\pi\sqrt{\frac{x}{a}}}$$

5. Frequency (f): It is given by

$$f = \frac{1}{T} \qquad \boxed{f = \frac{1}{2\pi}\sqrt{\frac{a}{x}}}$$

6. Phase: It is given by $(\omega t + \phi)$

7. Phase constant: It is given by ϕ

Energy consideration in SHM:

A particle executing SHM has two types of energy, viz.

(i) Kinetic Energy

(ii) Potential Energy

Kinetic energy arises due to its motion about the mean position and is given by

$$K.E. = \frac{1}{2}mv^2$$

where m is the mass of the particle and v is its velocity

Continued from SHM_2

S.H.M_3

Contd. from energy……

Now the velocity is given by

$$v = -\omega x_m \sin(\omega t + \phi)$$

So,

$$\boxed{K.E. = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi)} \quad (1)$$

Now the Potential energy is given by

$$\mathbf{P.E = \frac{1}{2}kx^2}$$

where $x = x_m \cos(\omega t + \phi)$

So $P.E. = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi)$

Also since $\omega = \sqrt{\frac{k}{m}}$ (angular velocity)

Or, $k = m\omega^2$

Then,
$$P.E. = \frac{1}{2}m\omega^2 x_m^2\cos^2(\omega t + \phi) \qquad (2)$$

Now, the total energy is defined as the sum of the kinetic and potential energy.

Then,

Total Energy (T.E.) = K.E. + P.E.

$$= \frac{1}{2} m\omega^2 x_m^2 [\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

1

$$\boxed{T.E. = \frac{1}{2} m\omega^2 x_m^2}$$

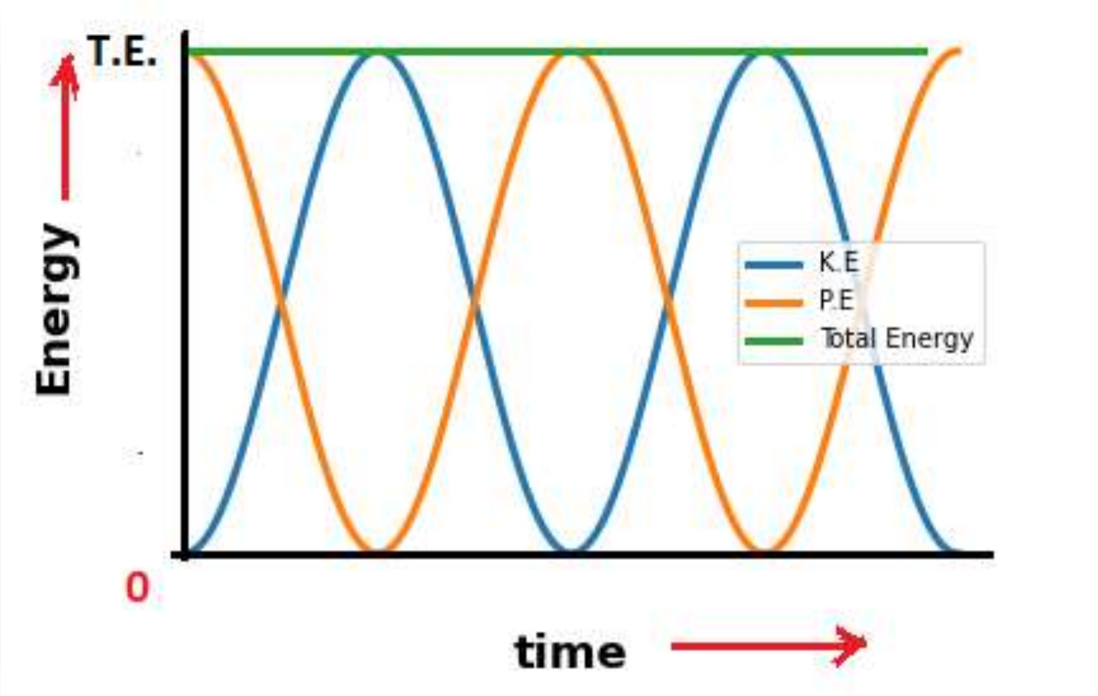
k

Or,

$$\boxed{T.E. = \frac{1}{2} k x_m^2}$$

This shows that the total energy remains conserved or constant although the kinetic and the potential energies vary with time.

Variation of Energy with time:



Variation of Energy with displacement:

?

H.W.

Homework/Assignment (SET 1):

1. Draw the graph of variation of Energy versus Displacement.
2. Draw the graphs of displacement vs time, velocity vs time and acceleration vs time.
3. What is S.H.M? Write the differential equation of S.H.M.
4. What is S.H.M? Discuss the characteristics of S.H.M with neat graphs. Also discuss the energy consideration in S.H.M with graphs. (9 marks/Long Q)

Numericals:

1. A body of mass 0.3 kg executes SHM with a period of 2.5 sec and amplitude of 4 cm. Calculate the amplitude, velocity, acceleration and kinetic energy.

gET YOUR BRAIN EXERCISED !

Sol^n:

Use this format to do numericals

Here, given:

Mass of the body (m) = 0.3 kg

Period of oscillation (T) = 2.5 sec

Amplitude (x_m) = 4 cm = 0.04 m

Now,

(i) Max. velocity, v_{max} = ωx_m = $(\frac{2\pi}{T})x_m = \frac{2(3.14)(0.04)}{2.5}$

= 0.1 m/s

(ii) Max. Acceleration, a_{max} = $\omega^2 x_m = (\frac{2\pi}{T})^2 x_m$

= 0.252 m/s^2

(iii) Maximum kinetic energy, $K.E_{max} = \frac{1}{2}mv_{max}^2$

$$= \frac{1}{2}0.3(0.1)^2$$

$$= 1.5 \times 10^{-3} J$$

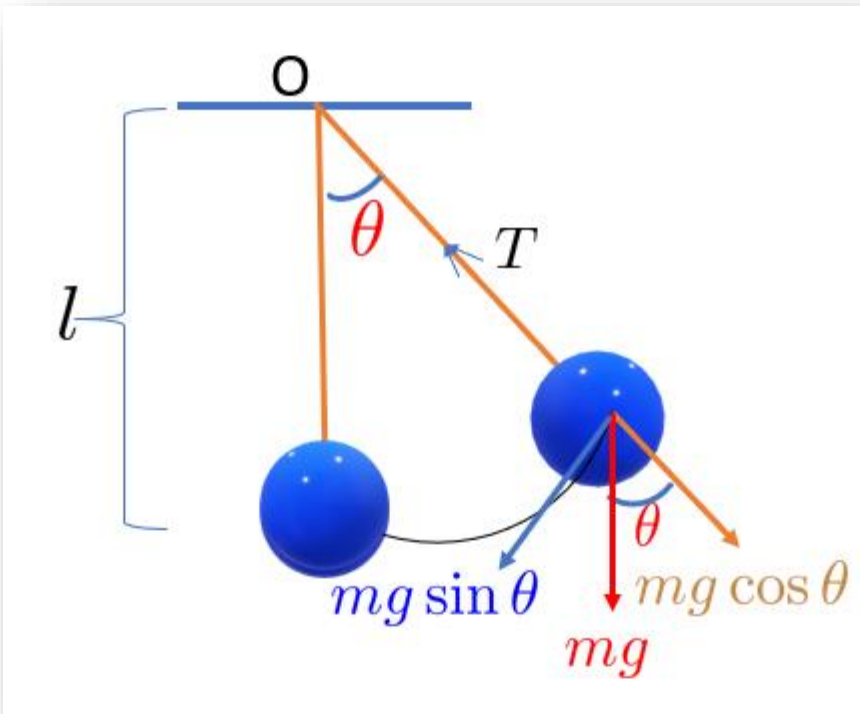
Assignment:

2. A small body of mass 0.1 kg is undergoing a SHM of amplitude 0.1 m and period 2 sec. (i) What is the maximum force on body? (ii) If the oscillations are produced in the spring, what should be the force constant?

Assignment:

3. When the displacement is one-half the amplitude, what fraction of the total energy is the K.E. and what fraction is P.E. in S.H.M? At what displacement is the energy half K.E. and half P.E.?

Simple Pendulum:



A simple pendulum consists of a metallic bob suspended with an extensible thread or rope at the point O. It is displaced through a small angle θ .

The tension T balances the component mg cosθ while mg sinθ provides the necessary restoring force F.

Then,

Restoring force (F) = -mg sinϴ   *(The negative sign is due to the opposite direction of the restoring force and the displacement.)*

Or, ma = -mg sinϴ

Or, a = -g sinϴ

For small angle ϴ, sinϴ≈ϴ

Or, a = -gϴ

Or, a = -g $\left(\frac{x}{l}\right)$ *, where* x is the small linear displacement.

Or, a + $\left(\frac{g}{l}\right)x$ *= 0*

which can be written as

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Where the angular frequency is given by

$$\omega = \sqrt{\frac{g}{l}}$$

The former equation is the differential equation of SHM. Hence the motion of simple pendulum is simple harmonic.

To find the time period (T):

Since the angular frequency of the simple pendulum is given by

$$\omega = \sqrt{\frac{g}{l}}$$

Also, the time period is given by

$$T = \frac{2\pi}{\omega}$$

So, the time period of the simple pendulum is given by

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Note:

(i) T is independent of the mass (*m*) of the pendulum and the angular displacement (θ).

(ii) T solely depends upon the distance between the point of suspension and the center of gravity (C.G) of the pendulum.

(iii) If θ is relatively large, the approximation $\sin\theta \approx \theta$ is no longer valid, hence the motion is no longer simple harmonic. T depends upon θ

and this type of motion is called anharmonic motion.

- Remember that we come across the same idea in Compound Pendulum. So in dealing with the theory of Compound Pendulum you are advised to through the theory of Simple Pendulum.

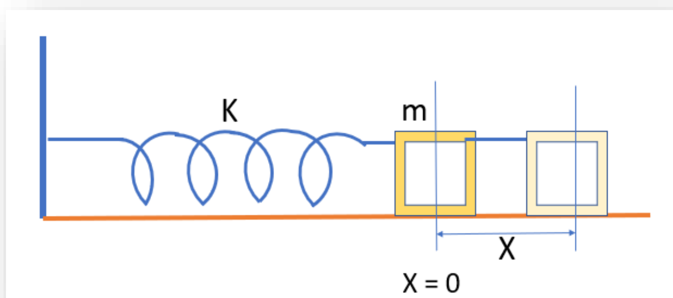
Mass-Spring System:

It consists of a block of mass m and a spring of spring constant *k*. *It has two types:*

(i) Horizontal Mass-Spring System
(ii) Vertical Mass-Spring System

(i) Horizontal Mass-Spring System:

Consider a horizontal mass-spring system of a block of mass m and a spring of spring constant k, as shown in the figure below.



Let the system be displaced through a distance X. (***Note***: You are also free to use small letter ‘x’ for displacement for consistency through the chapter of SHM). Then a restoring force is developed such that the system has the tendency to come back to the mean position.

It is found that the restoring force is directly proportional to the displacement from the mean position and is opposite to it .

i.e. $F \propto X$

or, $F = -kX$, k being a constant of proportion, and is called the spring constant.

or, $ma = -kX$

Substituting the value of acceleration, we get

$$m\frac{d^2X}{dt^2} + kX = 0$$

$$\boxed{\frac{d^2X}{dt^2} + \omega^2 X = 0} \qquad (1)$$

$$where, \omega = \sqrt{\frac{k}{m}}$$

(the angular frequency)

Equation (1) is the required differential equation of SHM. Hence the motion of the horizontal mass-spring system is simple harmonic.

To find the time period:

Since the time period is given by

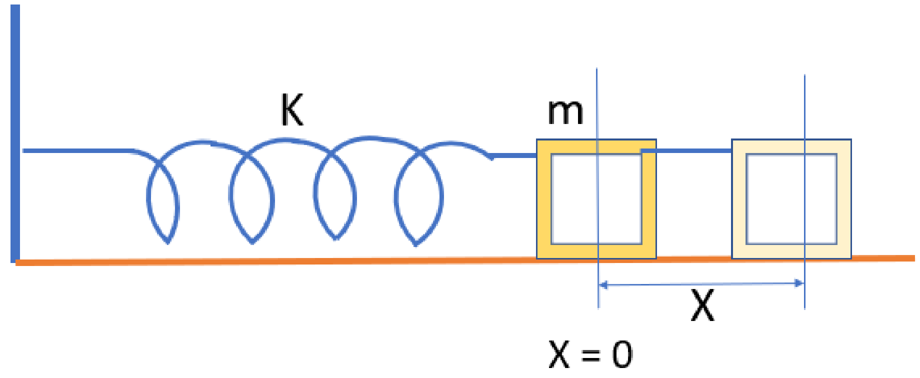
$$T = \frac{2\pi}{\omega}$$

And from above, we have

$$\omega = \sqrt{\frac{k}{m}}$$

So,

$$T = 2\pi\sqrt{\frac{m}{k}}$$

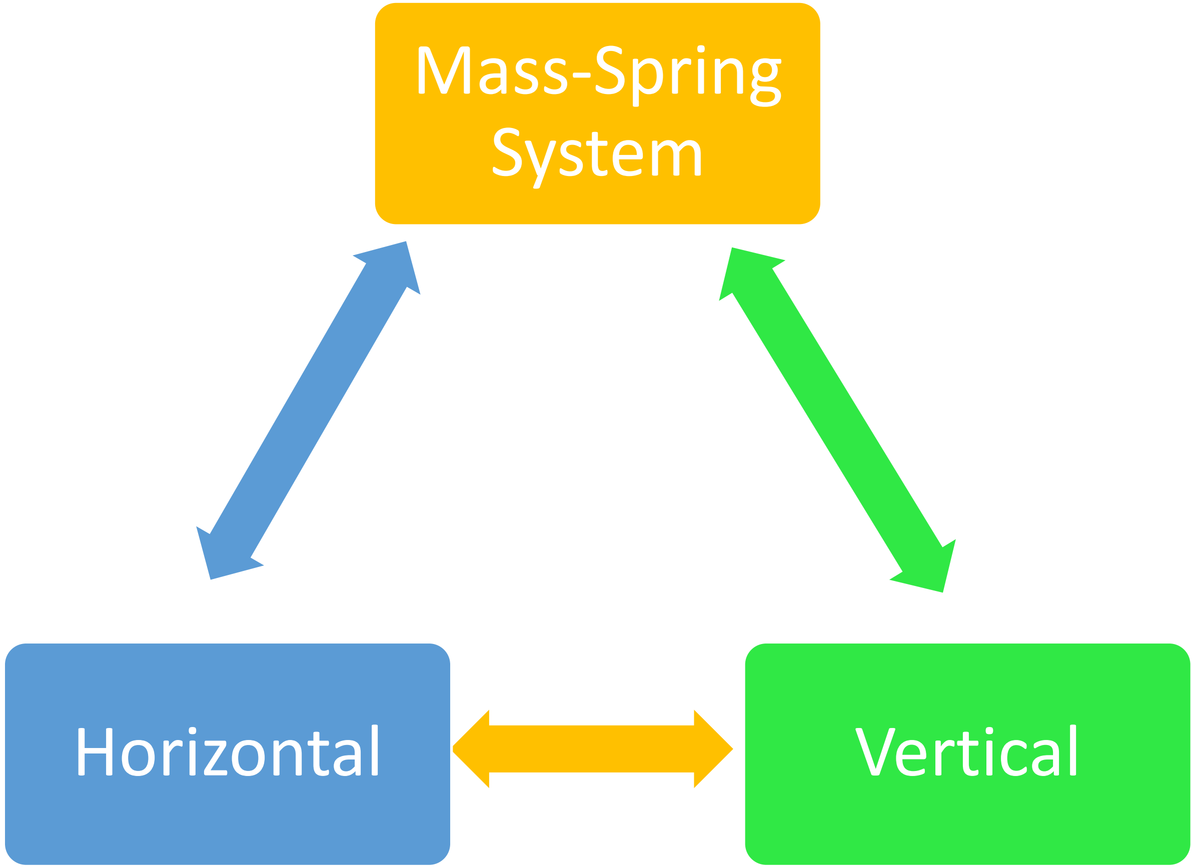


Mass-Spring System:

Contd…from Simple Pendulum

Mass-Spring System:

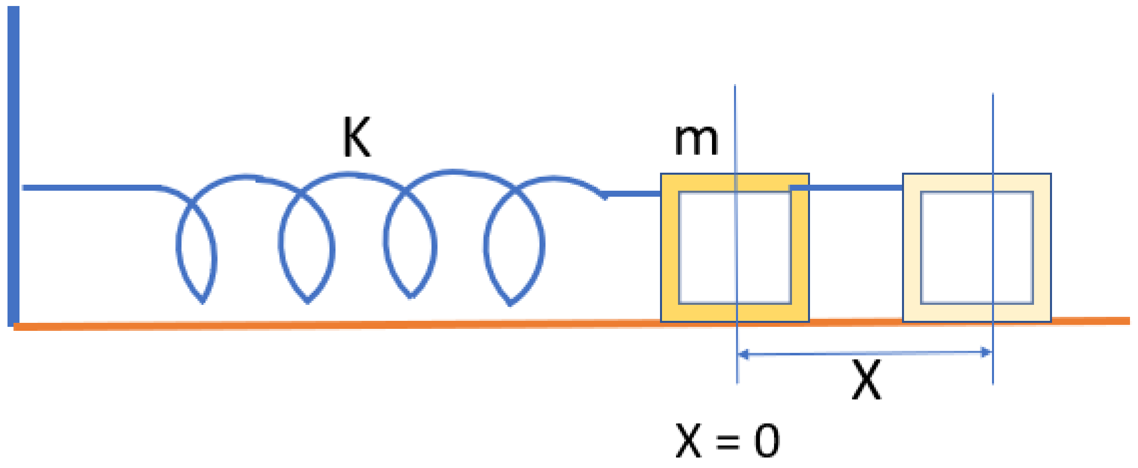
It consists of a block of mass m and a spring of spring constant k.



It has two types:

(i) Horizontal Mass-Spring System

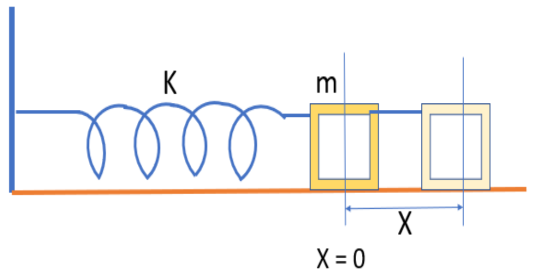
(ii) Vertical Mass-Spring System



(i) Horizontal Mass-Spring System:

Consider a horizontal mass-spring system of a block of mass m and a spring of spring constant k, as shown in the figure below.

Let the system be displaced through a distance X. Then a restoring force is developed such that the system has the tendency to come back to the mean position.



1st task: To show motion is SHM

F ∝ X

F = - k X

ma = - k X

$$\frac{d^2X}{dt^2} + \omega^2 X = 0$$

Diff. equation of SHM

So motion of Mass-spring System is Simple Harmonic.

2nd task: To find T

$$\omega = \sqrt{\frac{k}{m}}$$

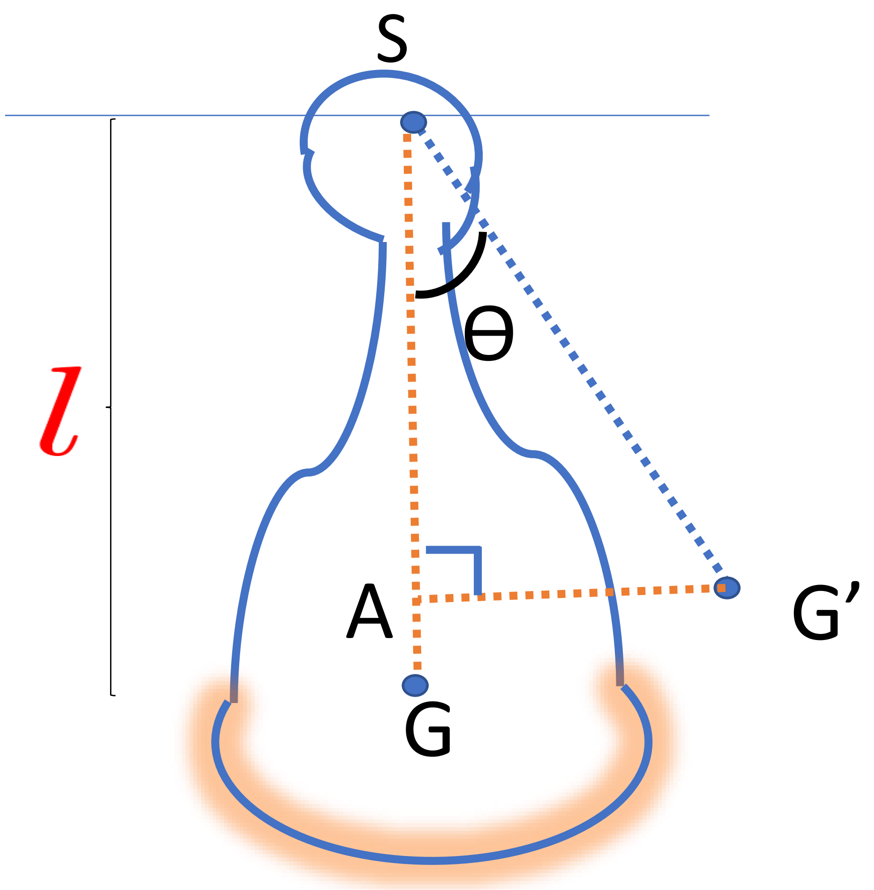
$$T = \frac{2\pi}{\omega}$$

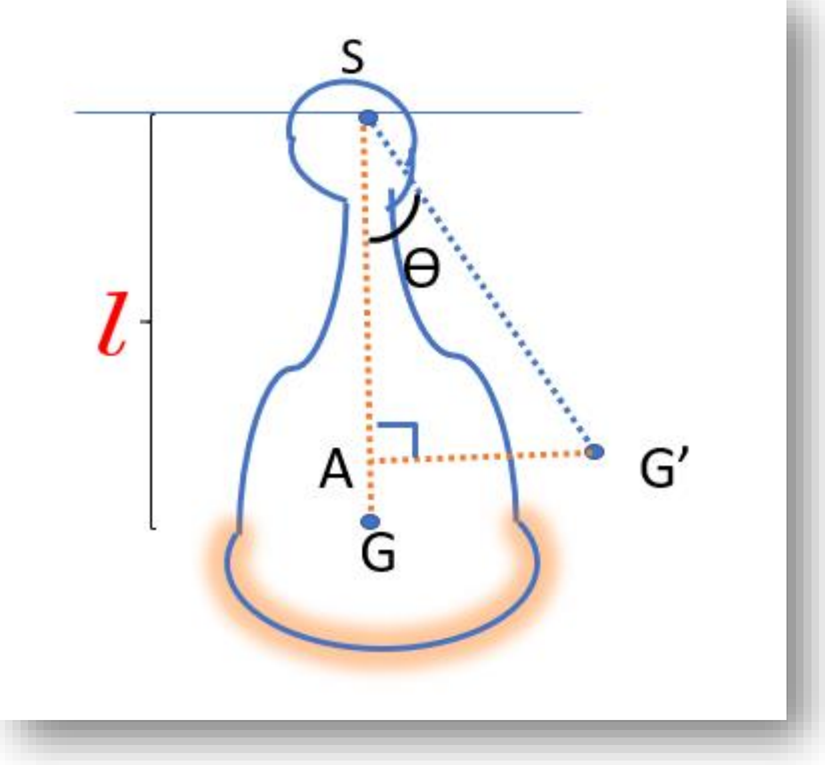
$$T = 2\pi\sqrt{\frac{m}{k}}$$

Assignment:

Vertical Mass-spring System

Compound /Real/Physical Pendulum :





Restoring torque is given by

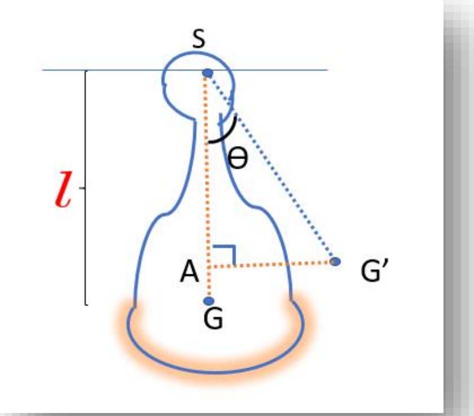
$$\tau = Force.Perpendiculardistance$$

$$I\alpha = mg.G'A$$

$$I\alpha = mg.l\sin\theta$$

Also, $I = I_{CG} + ml^2$

$$I = mk^2 + ml^2$$



$$\tau = I\alpha = -mgl\sin\theta$$

$$\alpha = \frac{d^2\theta}{dt^2}$$

Angular acceleration

Then,

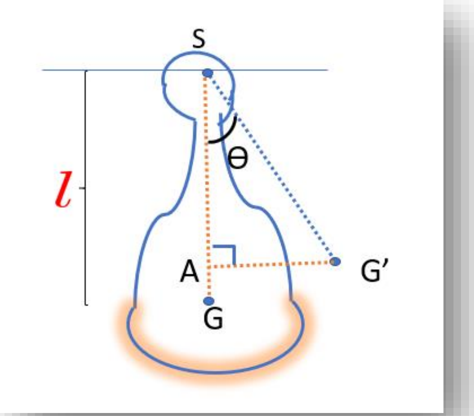
$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

$$\sin\theta \approx \theta$$

So, $$\frac{d^2\theta}{dt^2} + \left(\frac{mgl}{I}\right)\theta = 0$$

$$\omega^2$$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$



$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

This is the differential equation of SHM. Hence the motion of Compound Pendulum is simple Harmonic

1st task is done.
Mission accomplished!

2nd task: To find T

So, $\frac{d^2\theta}{dt^2} + (\frac{mgl}{I})\theta = 0$

ω^2

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

Previous slide

$$\omega = \sqrt{\frac{mgl}{I}}$$

This is the angular frequency

$$\omega = \sqrt{\frac{mgl}{I}}$$ → **Previous slide**

So, $$T = \frac{2\pi}{\omega}$$

Combining these two equations,

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

Also, $I = I_{CG} + ml^2$

$I = mk^2 + ml^2$ → **Previous slide**

So,

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

Dividing by l

$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

Very very important

Since for the compound pendulum, T is given by

$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

And for the simple pendulum, T is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Let $L = l + k^2/l$

called the length of the equivalent simple pendulum or the equivalent length of the simple pendulum or the reduced length of the compound pendulum.

Since for the compound pendulum, T is given by

$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

And for the simple pendulum, T is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Let $L = l + k^2/l$

called the length of the equivalent simple pendulum or the equivalent length of the simple pendulum or the reduced length of the compound pendulum.

Previous slide

Then,

$$T = 2\pi\sqrt{\frac{L}{g}}$$

This is the Time Period of the Compound Pendulum in terms of the length of the equivalent simple pendulum or the equivalent length of the simple pendulum or the reduced length of the compound pendulum.

Assignment:

Q. Define moment of inertia and radius of gyration. (2 marks)

Q. Show that the motion of compound Pendulum is Simple Harmonic. Hence find its time period. (9 marks)

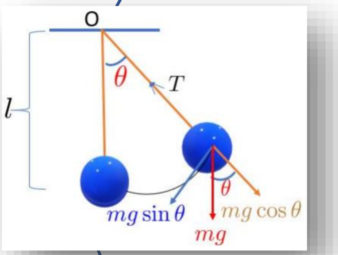
Or ⇕ Equivalent question

Q. Show that the motion of compound Pendulum is Simple Harmonic. Hence find its time period in terms of the length of the equivalent simple pendulum or the equivalent length of the simple pendulum or the reduced length of the compound pendulum.. (9 marks)

$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

For Compound Pendulum

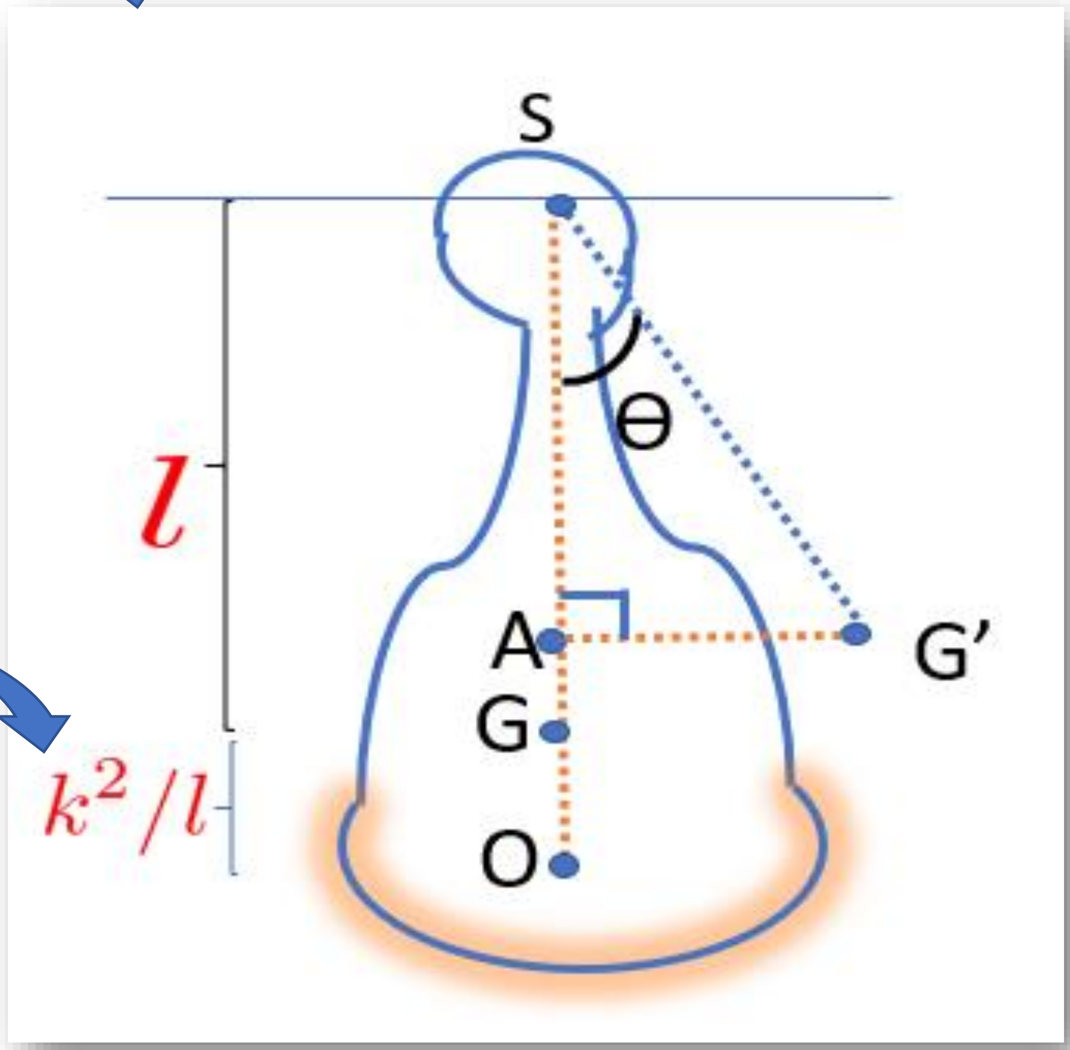
Previous Slide



$$T = 2\pi\sqrt{\frac{l}{g}}$$

Remember this for Simple Pendulum

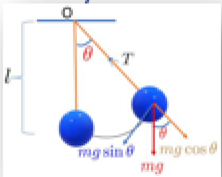
The factor k^2/l is the extra term in the Compound Pendulum



$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

For Compound Pendulum

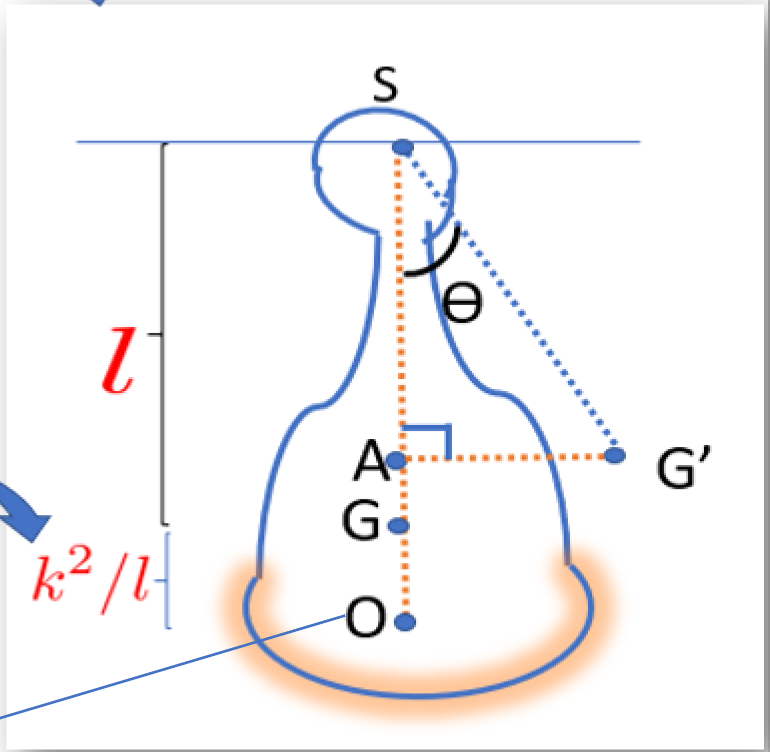
Previous Slide



$$T = 2\pi\sqrt{\frac{l}{g}}$$

Remember this for Simple Pendulum

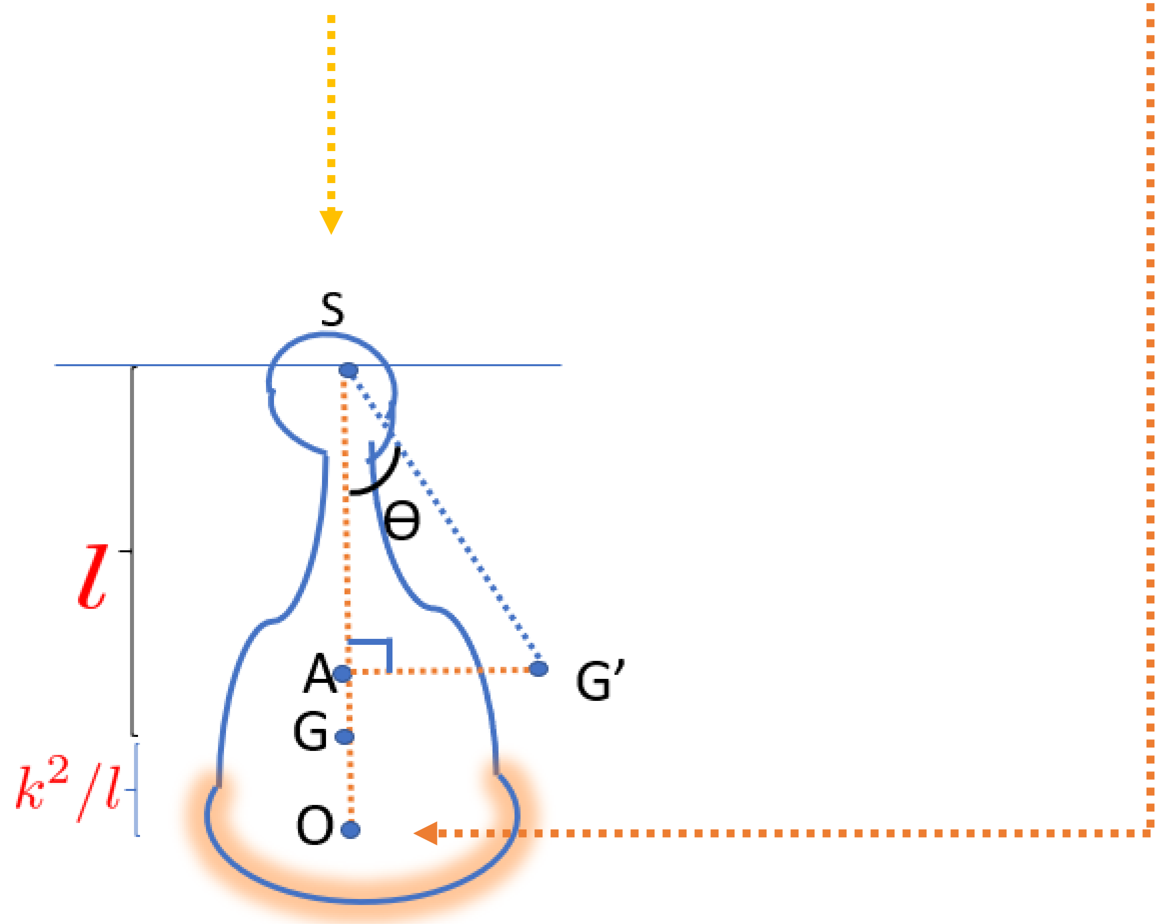
The factor k^2/l is the extra term in the Compound Pendulum



Previous Slide

This guy, THIS point which lies at a distance of k^2/l below CG is called the Point of Oscillation, & is denoted by the point 'O'

To show that the point of suspension and the point of oscillation are interchangeable:



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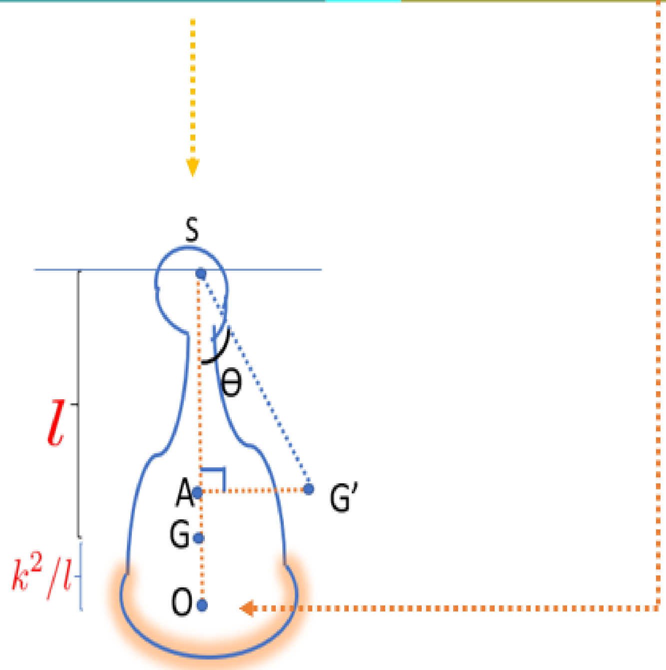
To show that the point of suspension and the point of oscillation are interchangeable:



For the Point of Suspension 'S', the time period is given by

$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

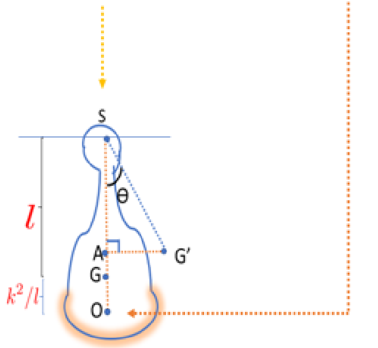
Let $k^2/l = l'$

Then,

$$T = 2\pi\sqrt{\frac{l + l'}{g}}$$

Contd…

To show that the point of suspension and the point of oscillation are interchangeable:



For the Point of Suspension 'S', the time period is given by

$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

Let $k^2/l = l'$

Then,

$$T = 2\pi\sqrt{\frac{l + l'}{g}}$$

Previous Slide

But

$$k^2/l = l' \rightarrow k^2/l' = l$$

So,

$$T' = 2\pi\sqrt{\frac{l' + l}{g}}$$

Now, the time period for the Point of Oscillation 'O' is given by

$$T' = 2\pi\sqrt{\frac{l' + k^2/l'}{g}}$$

So, $T = T'$

Hence, the point of suspension and the point of oscillation are interchangeable:

Note: $$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

T will be maximum (i.e. $T = \infty$) for

(i) $g = o$

(ii) $l = o$ → point of suspension is the C.G itself

(iii) $l = \infty$

Minimum Time Period: → V. Imp

Q. Show that the period of the Compound Pendulum is minimum at $l = k$

OR

The period is minimum when the point of suspension and the point of oscillation are equidistant from C.G.

Minimum Time Period: → V. imp

Q. Show that the period of the Compound Pendulum is minimum at $l = k$

OR

The period is minimum when the point of suspension and the point of oscillation are equidistant from C.G.

Solution: Since

$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

Squaring both sides,

$$T^2 = 4\pi^2\left(\frac{l + k^2/l}{g}\right)$$

Differentiating both sides, we get

$$\frac{dT^2}{dl} = \frac{dT^2}{dT}\frac{dT}{dl}$$

Or, $$2T\frac{dT}{dl} = \frac{4\pi^2}{g}\frac{d}{dl}\left(\frac{k^2}{l} + l\right)$$

$$= \frac{4\pi^2}{g}\left(k^2\frac{dl^{-1}}{dl} + \frac{dl}{dl}\right)$$

$$= \frac{4\pi^2}{g}\left(\frac{-k^2}{l^2} + 1\right)$$

Differentiating both sides, we get

$$\frac{dT^2}{dl} = \frac{dT^2}{dT}\frac{dT}{dl}$$

Or, $2T\frac{dT}{dl} = \frac{4\pi^2}{g}\frac{d}{dl}(\frac{k^2}{l} + l)$

$$= \frac{4\pi^2}{g}(k^2\frac{dl^{-1}}{dl} + \frac{dl}{dl})$$

$$= \frac{4\pi^2}{g}(\frac{-k^2}{l^2} + 1)$$

For T to be minimum or maximum:

$$\frac{dT}{dl} = 0$$

So,

$$0 = \frac{4\pi^2}{g}(\frac{-k^2}{l^2} + 1)$$

$$\Rightarrow 0 = \frac{-k^2}{l^2} + 1$$

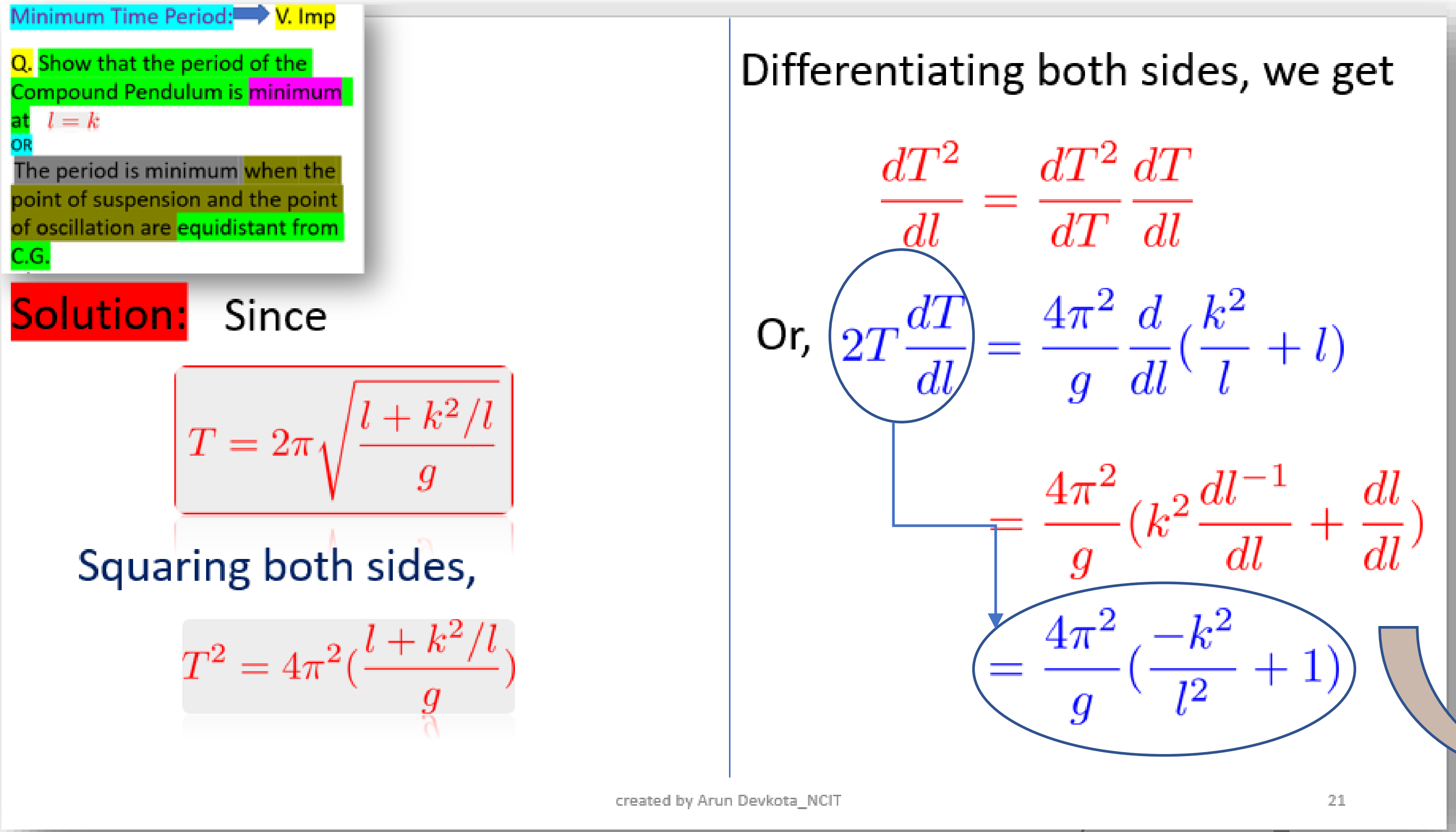
$$\Rightarrow \frac{k^2}{l^2} = 1$$

$$\Rightarrow k^2 = l^2$$

$$\Rightarrow \boxed{l = k}$$ (Taking the positive value only)

Hence for $l = k$, T will be maximum or minimum.

If we show $\frac{d^2T}{dl^2} > 0$, T will be minimum at $l = k$



Minimum Time Period: → V. Imp

Q. Show that the period of the Compound Pendulum is minimum at $l = k$

OR

The period is minimum when the point of suspension and the point of oscillation are equidistant from C.G.

Solution: Since

$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

Squaring both sides,

$$T^2 = 4\pi^2\left(\frac{l + k^2/l}{g}\right)$$

Differentiating both sides, we get

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$$= \frac{4\pi^2}{g}\left(k^2\frac{dl^{-1}}{dl} + \frac{dl}{dl}\right)$$

$$= \frac{4\pi^2}{g}\left(\frac{-k^2}{l^2} + 1\right)$$

created by Arun Devkota_NCIT 21

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Differentiate this one w.r.t. 'l'

2 is cancelled out from LHS and RHS

$$\left(\frac{dT}{dl}.\frac{dT}{dl} + T\frac{d^2T}{dl^2}\right) = \frac{2\pi^2}{g}\frac{d}{dl}\left(\frac{-k^2}{l^2} + 1\right)$$

$$\left[\left(\frac{dT}{dl}\right)^2 + T\frac{d^2T}{dl^2}\right] = \frac{2\pi^2}{g}\frac{d}{dl}\left(\frac{-k^2}{l^2} + 1\right)$$

$$\left[\left(\frac{dT}{dl}\right)^2 + T\frac{d^2T}{dl^2}\right] = \frac{2\pi^2}{g}\left[(-k^2)(-2)l^{-3}\right]$$

Since $\frac{dT}{dl} = 0$

Or, $T\frac{d^2T}{dl^2} = \frac{4\pi^2}{g}\frac{k^2}{l^3}$

Since RHS>0 and T>0,

$$\frac{d^2T}{dl^2} > 0$$

Hence the period of the Compound Pendulum is minimum at $l = k$

Again, since at $\boxed{l = k}$, period is minimum, the minimum period is given by

$$T_{min} = 2\pi\sqrt{\frac{k + k^2/k}{g}}$$

$$T_{min} = 2\pi\sqrt{\frac{k + k}{g}}$$

$$T_{min} = 2\pi\sqrt{\frac{2k}{g}}$$

So, The period is minimum when the point of suspension and the point of oscillation are equidistant from C.G.

Compound Pendulum contd…

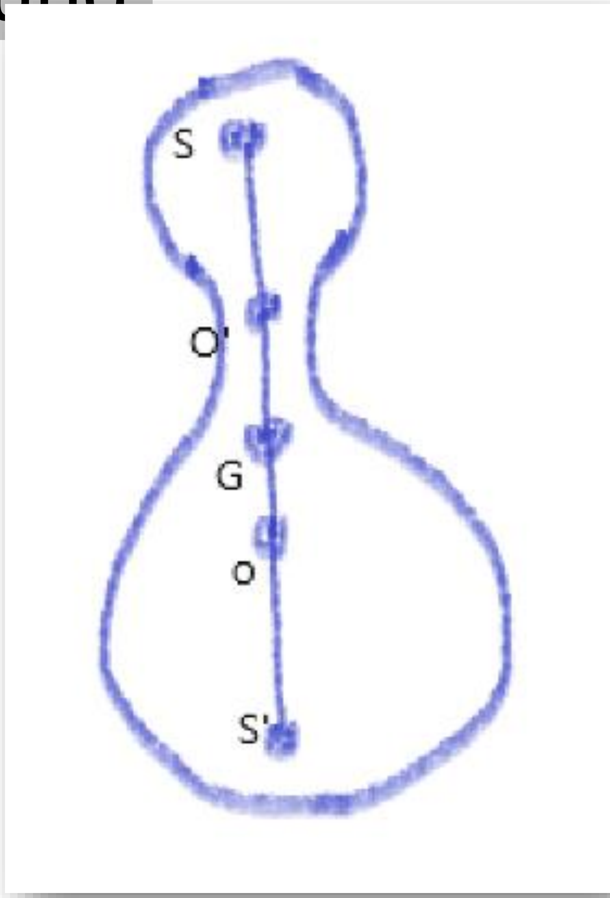
+

Bar Pendulum

Question:

Show that there are 4 colinear points in the compound pendulum for which the period is same.

Soln:



Let the pendulum be given as shown. S and O are the point of suspension and point of oscillation at distances l and k^2/l from the C.G., denoted by G.

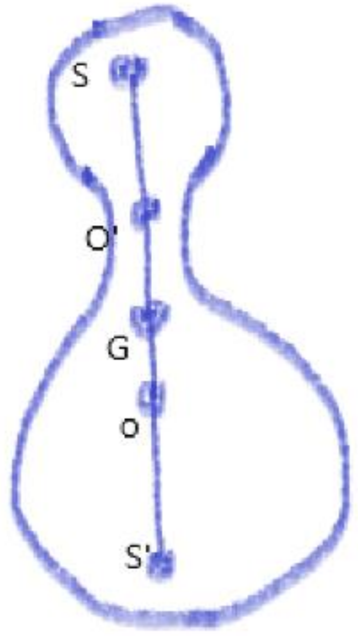
Now, take radius equal to GO = k^2/l and cut the vertical axis above C.G at O' such that GO' =$k^2/l = GO$

Then the time period remains same for the points O and O’

Again , Now, take radius equal to
SG = l and cut the vertical axis below C.G at O’ such that S’G = $l = SG$.

Again there are two points for which the period is same.

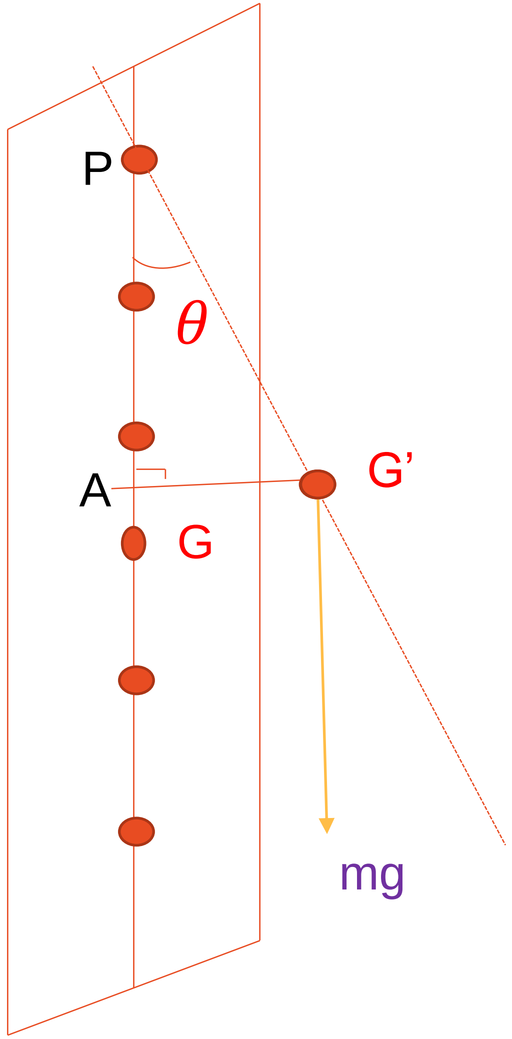
Thus in total there are 4 colinear points S, O, S’ and O’ for which the period is same.



Bar Pendulum:

Consider a bar Pendulum having C.G. at the point G and suspended at the point P.
It is displaced through a small angle θ.
Due to the restoring torque, it comes back to the mean position.
Now the restoring torque is given by

$$\tau = \text{Force . Perpendicular distance}$$



i.e. $\tau = mg.l\sin\theta$

But, $\tau = I\alpha = -mgl\sin\theta$

Now, $I\alpha = -mgl\sin\theta$

or, $I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

This is the differential equation of SHM. Hence the motion of the Bar Pendulum is Simple Harmonic

Here, $\omega^2 = \frac{mgl}{I}$

$$\omega = \sqrt{\frac{mgl}{I}}$$

called the angular frequency

Now,

$$T = \frac{2\pi}{\omega}$$ is the time period

So,

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

Also, $I = I_{CG} + ml^2$

$$I = mk^2 + ml^2$$

$$T = 2\pi\sqrt{\frac{l + k^2/l}{g}}$$

Now,

Squaring both sides,

$$T^2 = (4\pi^2/g)(\frac{k^2}{l} + l)$$

Or, $T^2 l = (4\pi^2/g)l^2 + (4\pi^2/g)k^2$

Or, $(\frac{4\pi^2}{g})l^2 + (-T^2)l + \frac{4\pi^2}{g}k^2 = 0$

which is quadratic in l

If l_1 and l_2 are the roots of the above equation,

Then,

$$l_1 + l_2 = -\frac{-T^2}{(4\pi^2)/g}$$

Or, $L = \frac{gT^2}{4\pi^2}$

Therefore, $g = \frac{4\pi^2 L}{T^2}$

And

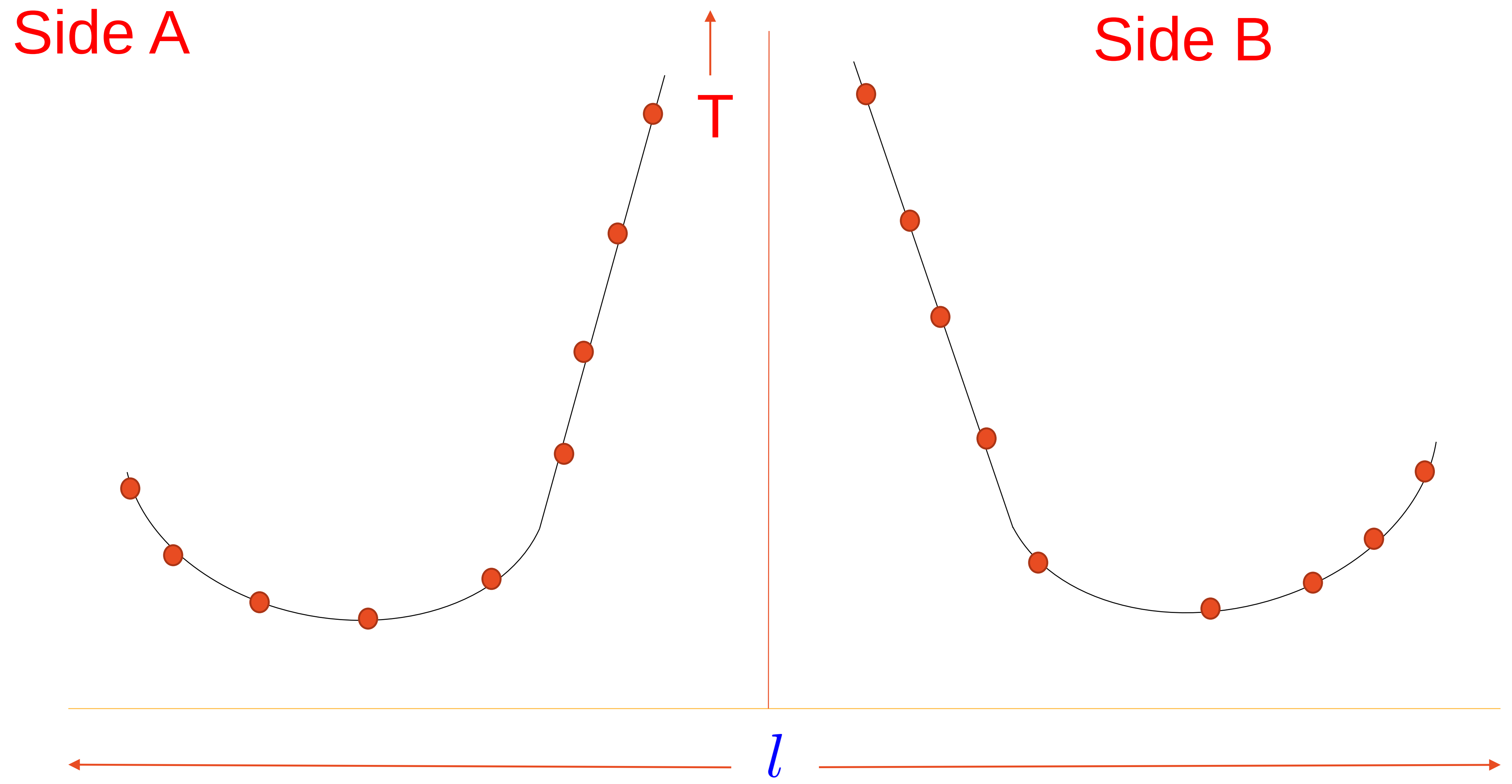
$$l_1.l_2 = \frac{(4\pi^2/g)k^2}{4\pi^2/g}$$

Or, $k = \sqrt{l_1.l_2}$

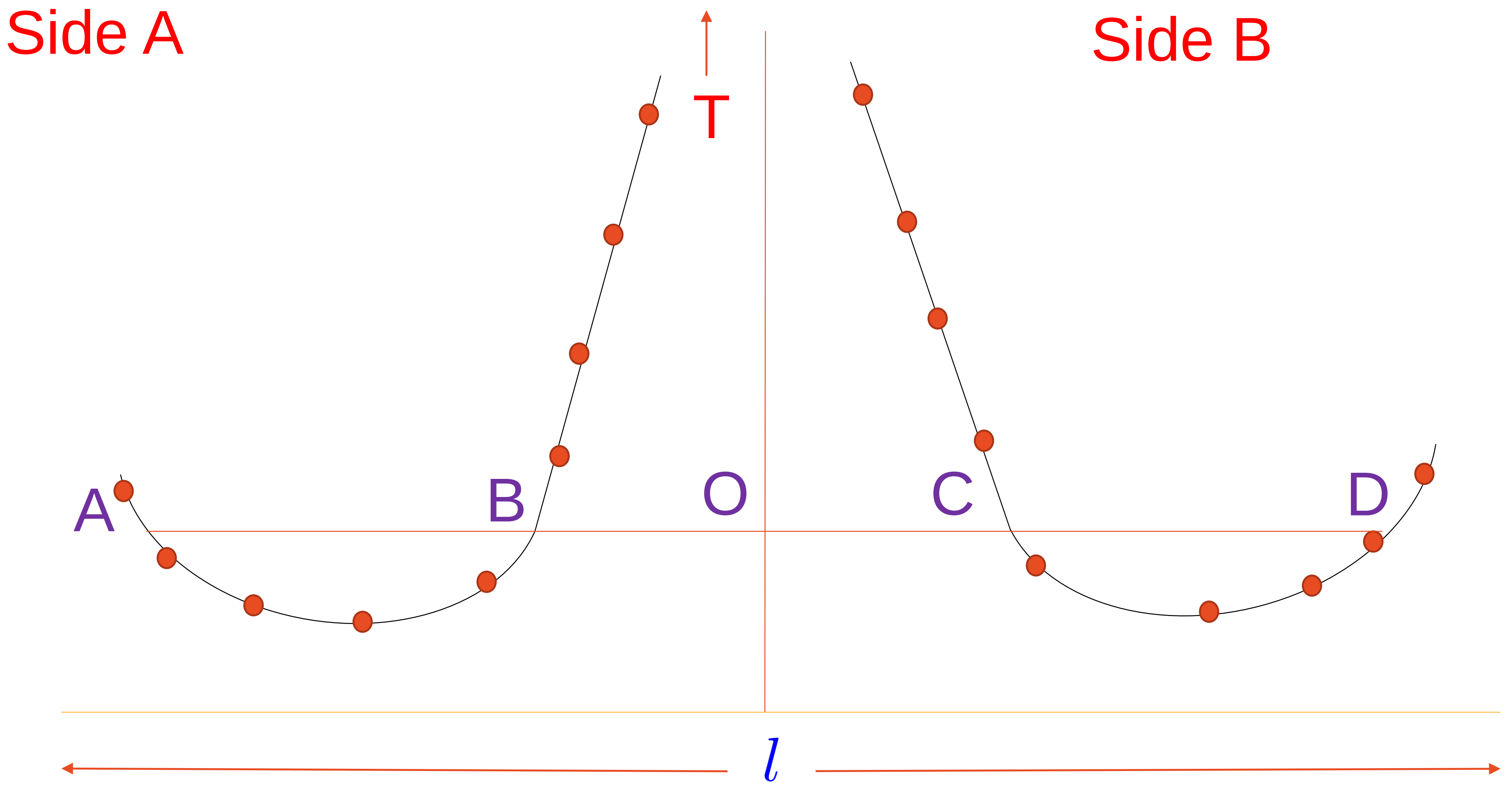
Thus knowing the value of l_1, l_2 and T,
We can determine the value of g and k.

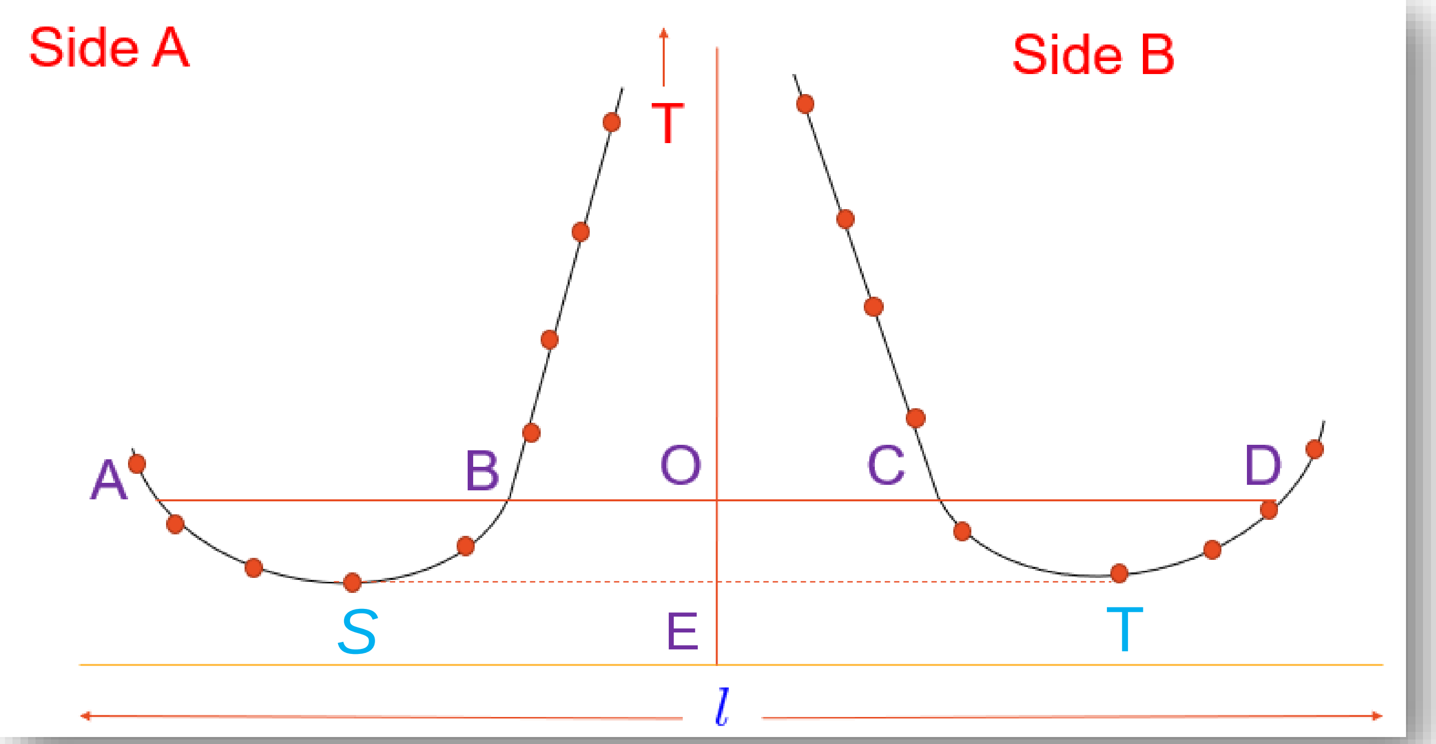
1st method to find g & k:

We plot a graph of T vs l and get curves as shown below .



Now, we draw straight lines like ABOCD.





In the graph, $OA = l_1$ & $OC = l_2$

$OD = l_1$ & $OB = l_2$

Thus, $L = AC$ or $L = BD$

& $OE = T$

In this way, the value of g & k are determined using the formulas

$$g = \frac{4\pi^2 L}{T^2} \quad \& \quad k = \sqrt{l_1 . l_2}$$

Notes:

(i) If A is the point of suspension, C is the point of oscillation. Similarly, if D is the point of suspension, B is the corresponding point of oscillation. # Here, A & C or B & D lie on the opposite sides of C.G.

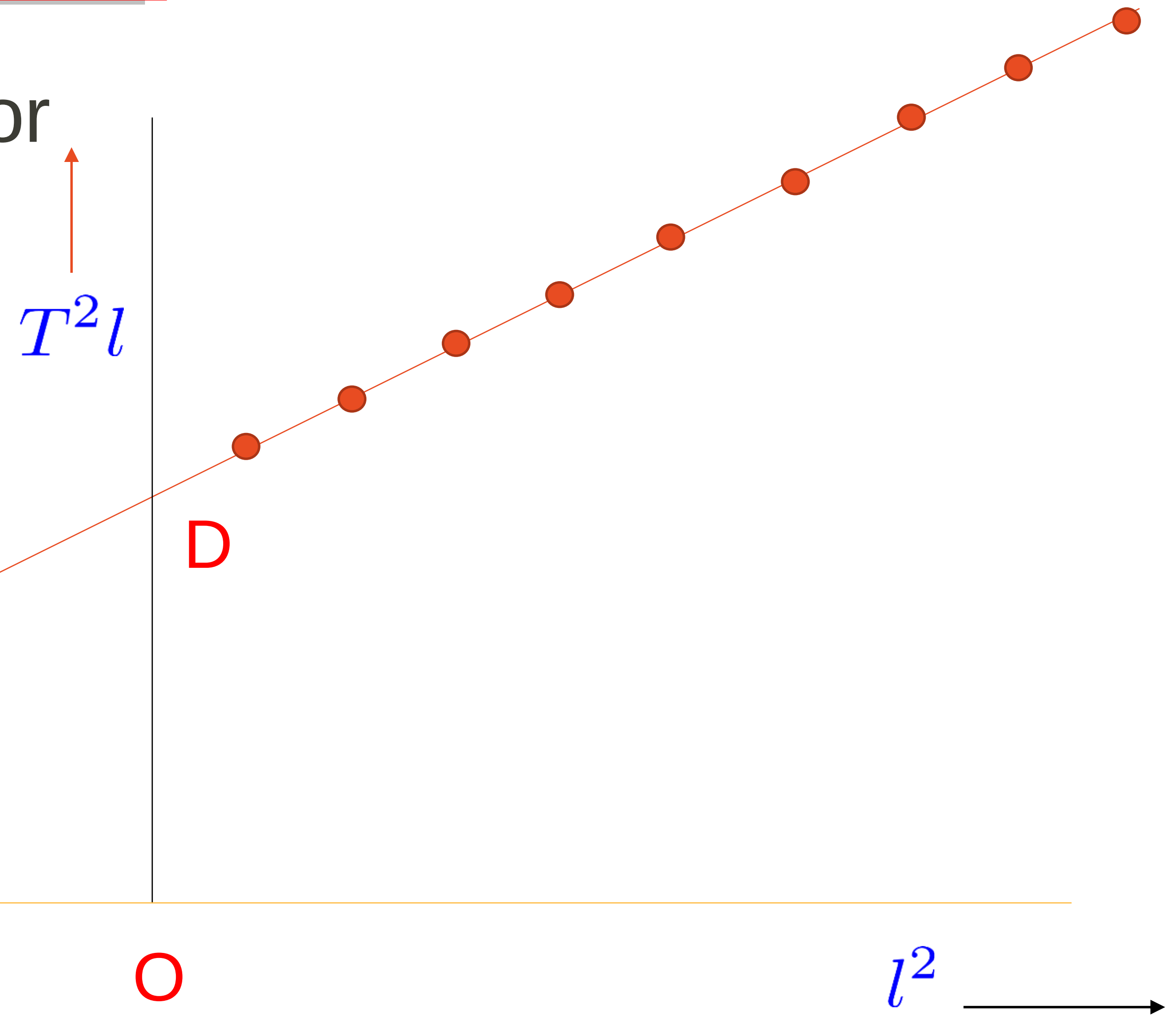
(ii) L = AC or BD is the length of the equivalent simple pendulum or the reduced length of the compound pendulum.

(iii) There are 4 colinear points: A,B,C & D for which the period(T = OE) is same.

(iv) Here, the points S & T are the points for which the period is minimum correspond to l=k & are equidistant from C.G.

Alternative method to find g and k:

Plot the graph of $T^2 l$ vs. l^2 for both sides A & B separately.



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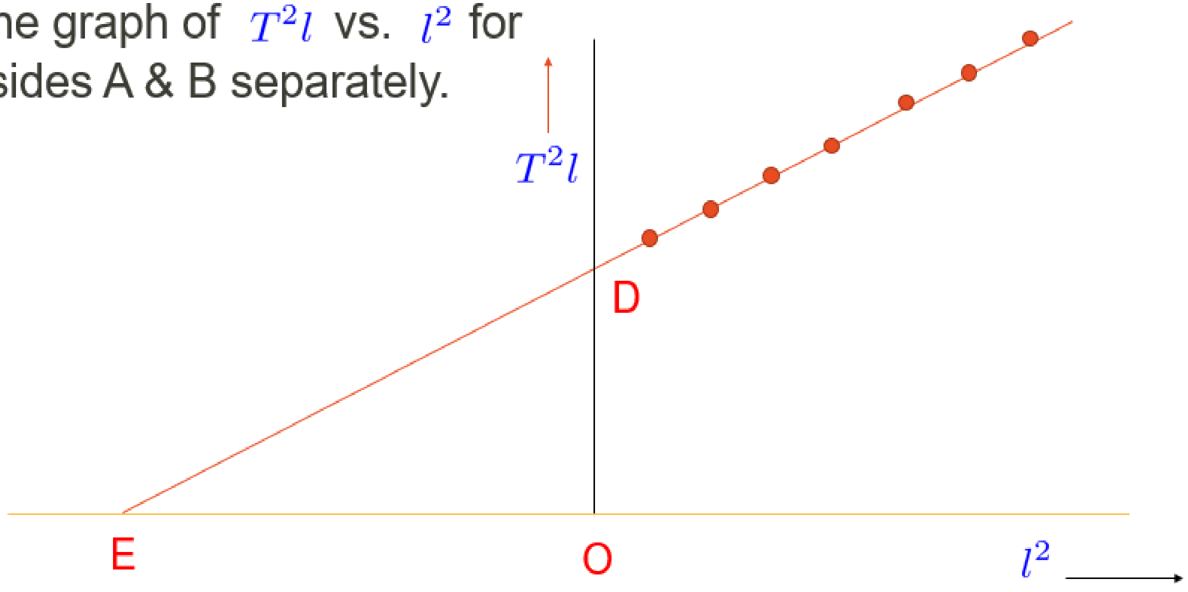


Then, from previous equation,

$$T^2 l = (\frac{4\pi^2}{g}) l^2 + \frac{4\pi^2}{g} k^2$$

Comparing above *equation with* $y = mx + c$, *we have*

$$y = T^2 l$$

$$x = l^2$$

$$m = \frac{4\pi^2}{g}$$

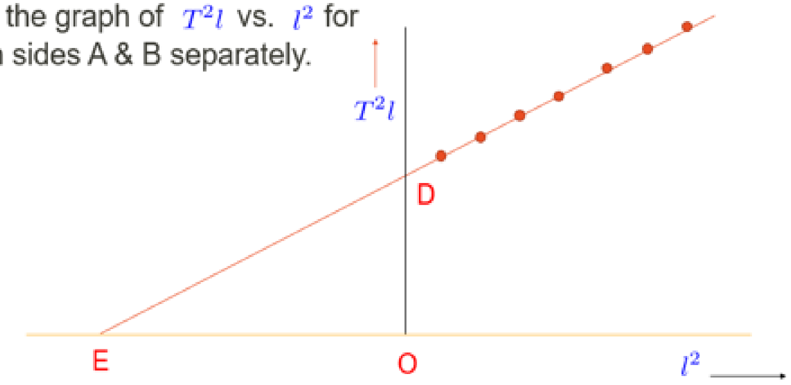
$$c = \frac{4\pi^2}{g} k^2$$

From graph,

$$m = \frac{OD}{OE}, \quad c = OD$$

Alternative method to find g and k:

Plot the graph of $T^2 l$ vs. l^2 for both sides A & B separately.



Then, from previous equation,

$$T^2 l = \left(\frac{4\pi^2}{g}\right) l^2 + \frac{4\pi^2}{g} k^2$$

Comparing above equation with $y = mx + c$, *we have*

$$y = T^2 l$$

$$x = l^2$$

$$m = \frac{4\pi^2}{g}$$

$$c = \frac{4\pi^2}{g} k^2$$

From graph,

$$m = \frac{OD}{OE}, \quad c = OD$$

Thus, the values of OD & OE are determined from graph, hence the slope (m) is determined which helps to find the value of g. Further the y-intercept (c) and the value of g help to find the value of k.

Question:

Derive the non-differential form of SHM.

Hint:

In the beginning of the chapter we defined SHM.

$$F \propto x$$

Then, $F = -kx$

And arrived at the differential equation of SHM

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

And wrote directly the solution of the differential equation above can be written as

$$x = x_m \cos(\omega t + \phi)$$

Now, we are arriving above equation step by step

Derive the non-differential form of SHM.

Solution:

The motion in which the restoring force is directly proportional to the displacement from the mean position and is opposite to it is called SHM.

i.e. $F \propto x$

$$F = -kx$$

$$ma = -kx$$

Since $a = \dfrac{d^2x}{dt^2}$

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

Let $\omega^2 = \dfrac{k}{m}$

Or, $\omega = \sqrt{\dfrac{k}{m}}$

(the angular frequency)

So,

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\frac{d}{dt}(\frac{dx}{dt}) = -\omega^2 x$$

Multiplying both sides by $\frac{dx}{dt}$

$$\frac{d}{dt}(\frac{dx}{dt})\frac{dx}{dt} = -\omega^2 x\frac{dx}{dt}$$

omit dt from both sides

Or, $\frac{dx}{dt}d(\frac{dx}{dt}) = -\omega^2 x dx$

Integrating both sides, we get

$$\frac{1}{2}(\frac{dx}{dt})^2 = -\frac{\omega^2 x^2}{2} + c \quad \text{(1)}$$

where C is the constant of integration.

So, $$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\frac{d}{dt}(\frac{dx}{dt}) = -\omega^2 x$$

Multiplying both sides by $\frac{dx}{dt}$

$$\frac{d}{dt}(\frac{dx}{dt})\frac{dx}{dt} = -\omega^2 x \frac{dx}{dt}$$

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Integrating both sides, we get

$$\frac{1}{2}(\frac{dx}{dt})^2 = -\frac{\omega^2 x^2}{2} + c \quad \text{——— (1)}$$

where C is the constant of integration.

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But $\frac{dx}{dt} = v$ is the velocity which is maximum at the mean position.

Then,

$$v_{max} = \omega x_m \quad \text{——— (2)}$$

From (1) & (2),

$$\frac{1}{2}v_{max}^2 = -\frac{\omega^2.0^2}{2} + c$$

$$\frac{1}{2}\omega^2 x_m^2 = c \quad \text{——— (3)}$$

So, $\frac{d^2x}{dt^2} = -\omega^2 x$

$\frac{d}{dt}(\frac{dx}{dt}) = -\omega^2 x$

Multiplying both sides by $\frac{dx}{dt}$

$\frac{d}{dt}(\frac{dx}{dt})\frac{dx}{dt} = -\omega^2 x \frac{dx}{dt}$

omit dt from both sides

Or, $\frac{dx}{dt} d(\frac{dx}{dt}) = -\omega^2 x dx$

Integrating both sides, we get

$\frac{1}{2}(\frac{dx}{dt})^2 = -\frac{\omega^2 x^2}{2} + c$ —— (1)

where C is the constant of integration.

Previous Slide

But $\frac{dx}{dt} = v$ is the velocity which is maximum at the mean position.

Then,

$v_{max} = \omega x_m$ —— (2)

From (1) & (2),

$$\frac{1}{2}v_{max}^2 = -\frac{\omega^2 .0^2}{2} + c$$

$\frac{1}{2}\omega^2 x_m^2 = c$ —— (3)

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So from (1) & (3),

$$\frac{1}{2}(\frac{dx}{dt})^2 = -\frac{\omega^2 x^2}{2} + \frac{1}{2}\omega^2 x_m^2$$

$$\frac{1}{2}(\frac{dx}{dt})^2 = \frac{1}{2}\omega^2 (x_m^2 - x^2)$$

$$(\frac{dx}{dt})^2 = \omega^2 (x_m^2 - x^2)$$

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$$\left(\frac{dx}{dt}\right)^2 = \omega^2(x_m^2 - x^2)$$

Taking square root, we have

$$\frac{dx}{dt} = \pm\omega\sqrt{x_m^2 - x^2}$$

Taking positive sign,

$$\frac{dx}{dt} = \omega\sqrt{x_m^2 - x^2}$$

or, $$\frac{dx}{\sqrt{x_m^2 - x^2}} = \omega dt$$

Integrating both sides, we have

$$\arcsin\frac{x}{x_m} = \omega t + \phi$$

$$x = x_m \sin(\omega t + \phi)$$

(4)

Again, taking negative sign,

$$\frac{dx}{dt} = -\omega\sqrt{x_m^2 - x^2}$$

Upon integrating, we get,

$$\arccos\frac{x}{x_m} = \omega t + \phi$$

$$x = x_m \cos(\omega t + \phi)$$

(5)

Equation (4) or (5) is the required non-differential form of SHM.

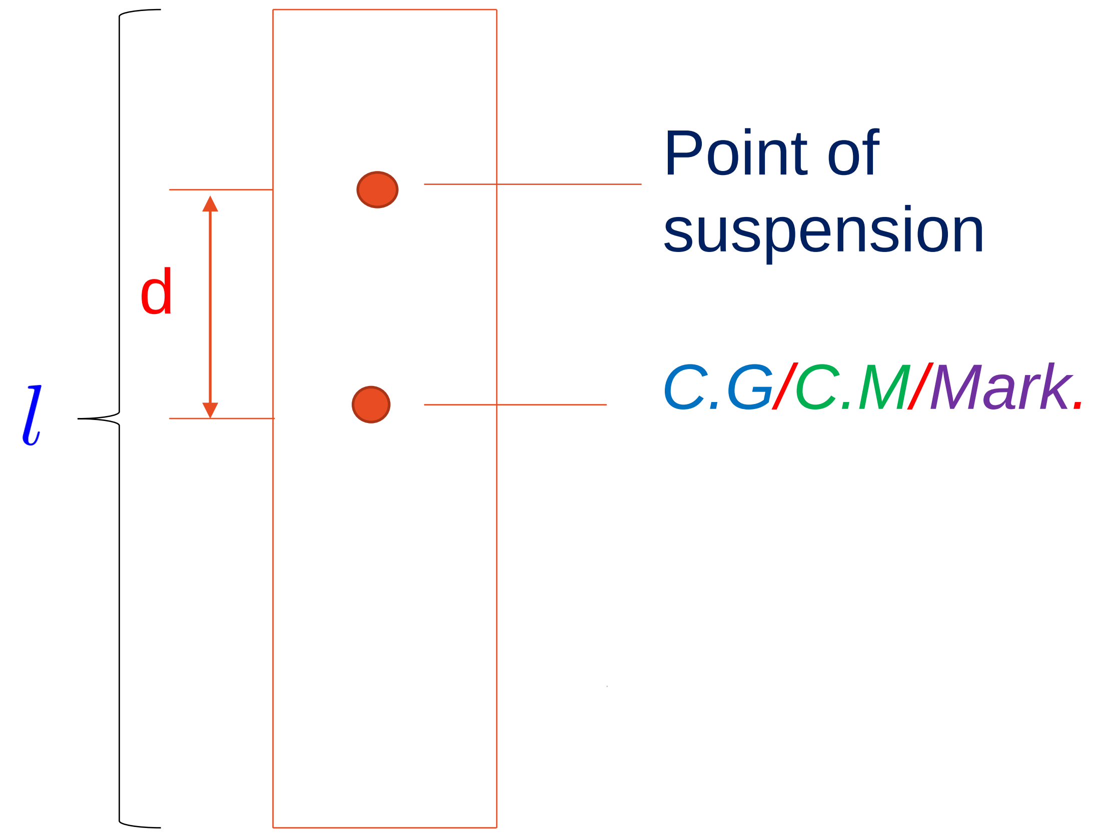
Numerical:

Q.1 Show that if a uniform stick of length ‘l’ is mounted so as to rotate about a horizontal axis perpendicular to the stick and at a distance ‘d’ from the center of mass or mark, period has a minimum value when $d = 0.289l$.

Solution:

The stick ,here, is a compound pendulum having total length l whose time period is given by

$$T = 2\pi\sqrt{\frac{\frac{k^2}{d} + d}{g}}$$



where

d is the distance between the point of suspension and the C.G.

& k is the radius of gyration

For the stick, the radius of gyration k is given by

$$k = \frac{TotalLength}{\sqrt{12}}$$

$$k = \frac{l}{\sqrt{12}}$$

$$k = \frac{1}{\sqrt{12}} l$$

$$k = 0.289l$$

For T to be minimum,

Distance between the point of suspension and C.G. = radius of gyration

$$d = k = \frac{l}{\sqrt{12}}$$

$$d = 0.289l$$

So, period has a minimum value when d = 0.289l.

Assignment:

Numerical:

Q.2

A small body of mass 0.1 kg is undergoing a SHM of amplitude 0.1 m and period 2 sec. (i) what is the maximum force on body? (ii) If the oscillations are produced in the spring, what should be the force constant?

Numerical:

Q.3

A meter stick suspended from one end swings as a physical pendulum (i) What is the period of oscillation? (ii) What would be the length of the simple pendulum that would have the same period?

Numerical:

Q.4

A physical pendulum consists of a meter stick that is pivoted at a small hole drilled through the stick at a distance x from mark. The period of oscillation is observed to be 2.5 s. Find the distance x. *(Ans: 5.57 cm)*

Numerical:

Q.5

What is the mechanical energy of the linear oscillator so that the initial position of the block is 11 cm at rest? The spring constant is 65 N/m. Calculate the K.E. and P.E. of the oscillator for its displacement being half of its amplitude.

Numerical:

Q.6 A uniform circular disc of radius R oscillates in a vertical plane about a horizontal axis. Find the distance of the axis of rotation from the center for which the period is minimum. What is the value of this period?

Ans: $\frac{R}{\sqrt{2}}$, $2\pi\sqrt{\frac{1.41R}{g}}$

Q.6

Show that the displacement equation represented by

$$x = a\sin(\omega t + \phi) + b\cos(\omega t + \phi)$$

or

$$x = a\sin\omega t + b\cos\omega t$$

represents S.H.M. or is a solution of S.H.M.